A SPECIAL CASE OF GENERALIZED HÖLDER FUNCTIONS

Maria Lupa

Institute of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
maria.lupa@im.pcz.pl

Abstract. In this paper some properties of a special case of generalized Hölder functions, which belong to the space $W_\gamma[a,b]$, are considered. These functions are *r*-times differentiable and their *r*-th derivatives satisfy the generalized Hölder condition. The main result of the paper is a proof of the theorem that product of two functions belonging to the space $W_\gamma[a,b]$ also belongs to this space.

Keywords: *Lipschitz condition, generalized Hölder condition, equivalency of norms*

Introduction

In the paper we introduce a function space $W_\gamma[a,b]$ and consider and prove some of its properties.

In the articles [1-4] the authors discussed Nemytskii operator determined on various function spaces (cf. also [5, 6]). For instance, it is shown there that in each function space $Lip[a,b], C^r[a,b], BV[a,b]$ a generating function of this operator exists and is affine with respect to the second variable. A similar result is obtained in [7] for the $W_\gamma[a,b]$-space. These results are then applied in [8, 9] to prove the existence and uniqueness of the solution of a certain functional equation in $W_\gamma[a,b]$.

The space W_γ and its properties

Let $[a,b]$ be a closed interval, where $a,b \in R$, $a<b$, $d \coloneqq b-a$. We assume that the following condition is fulfilled

$$(\Gamma)\ \gamma\colon [0,d] \to [0,\infty) \text{ is increasing and concave, } \gamma(0)=0,$$
$$\lim_{t\to 0^+}\gamma(t)=\gamma(0),\ \lim_{t\to d^-}\gamma(t)=\gamma(d).$$

Definition 1.

Denote by $W_\gamma[a,b]$ the set of all r-times differentiable functions, where $r \in N$, defined on the interval $[a,b]$ with values in R, such that their r-th derivatives satisfy the following condition: there exists a constant $M \geq 0$ such that

$$\left|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})\right| \leq M\gamma(|x - \bar{x}|),\ \bar{x}, x \in [a, b] \tag{1}$$

where γ fulfils condition (Γ).

Remark 1.

It is easily seen that $W_\gamma[a, b]$ contains the class of all r-times differentiable functions $\varphi: [a, b] \to R$, whose r-th derivatives satisfy the Lipschitz condition on $[a, b]$. This class is denoted by $LipC^r[a, b]$. Thus we have

$$LipC^r[a, b] \subset W_\gamma[a, b].$$

Remark 2.

Denote by $\gamma'_+(0)$ the right derivative of γ at $t = 0$. By (Γ) we have $0 \leq \gamma'_+(0) \leq +\infty$.

For $\varphi \in W_\gamma[a, b]$ and by the condition (Γ) we obtain

$$\left|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})\right| \leq M\gamma(|x - \bar{x}|) \leq M\gamma'_+(0)|x - \bar{x}|,\ \bar{x}, x \in [a, b]$$

i.e. $\varphi^{(r)}$ fulfils an ordinary *Lipschitz condition* with the constant $K = M\gamma'_+(0)$.

Thus if $\varphi \in W_\gamma[a, b]$ and $\gamma'_+(0)$ is finite then $\varphi \in LipC^r[a, b]$. In view of Remark 1 we get $LipC^r[a, b] = W_\gamma[a, b]$.

Therefore only the case $\gamma'_+(0) = +\infty$ is of interest.

Remark 3.

The functions of the form $\gamma(t) = t^\alpha$, where $0 < \alpha < 1$, $t \in [0, d]$, fulfil the assumption (Γ) and moreover $\gamma'_+(0) = +\infty$. Therefore the condition (1) is called *the generalized Hölder condition* or the $\gamma -$ *Hölder condition.*

Lemma 1.

If $\varphi \in W_\gamma[a, b]$ and $\gamma'_+(0) = +\infty$, then the functions $\varphi^{(k)}$, where $k = 0,1, \dots, r - 1$, fulfil the generalized Hölder condition with a function γ_0 and the constants $L_k, k = 0,1, \dots, r - 1$.

Proof.

By the condition $\gamma'_+(0) = +\infty$, we have $\gamma(t) \geq t$ in a right neighbourhood of zero. Suppose that $\gamma(d) \geq d$. Then we have $\gamma(t) \geq t$ for $t \in [0, d]$.

Let $\gamma(d) < d$. Define the function

$$\gamma_1(t) := \frac{d}{\gamma(d)}\gamma(t),\ t \in [0, d] \tag{2}$$

It is easily seen that γ_1 fulfils the condition (Γ) and $\gamma(t) \leq \gamma_1(t)$ for $t \in [0, d]$. Moreover, $\gamma_1(t) \geq t$ for $t \in [0, d]$. Clearly, if a function φ fulfils the generalized Hölder condition with the same function γ such that $\gamma(d) < d$ and with a constant M, then φ fulfils the generalized Hölder condition with the function γ_1, defined by (2), and the same constant M.
Define

$$\gamma_0 = \begin{cases} \gamma & \gamma(d) \geq d. \\ \gamma_1 & \gamma(d) < d \end{cases} \tag{3}$$

where γ_1 is defined by (2). Thus the functions $\varphi^{(k)}$, where $k = 0,1,\dots,r-1$, fulfil the generalized Hölder condition with the function γ_0 defined by (3), i.e. for $\bar{x}, x \in [a, b]$ we have

$$\left|\varphi^{(k)}(x) - \varphi^{(k)}(\bar{x})\right| \leq L_k|x - \bar{x}| \leq L_k\gamma_0(|x - \bar{x}|), k = 0,1,\dots,r-1.$$

This completes the proof.

Remark 4.

The space $W_\gamma[a, b]$ with the norm

$$\|\varphi\| := \sum_{k=0}^{r}\left|\varphi^{(k)}(a)\right| + sup\left\{\frac{\left|\varphi^{(r)}(x)-\varphi^{(r)}(\bar{x})\right|}{\gamma(|x-\bar{x}|)};\ x, \bar{x}\epsilon[a, b], x \neq \bar{x}\right\} \tag{4}$$

is a real normed vector space. Moreover, it is a Banach space ([5, 6]).

In the space $W_\gamma[a, b]$ we define the second norm by the formula

$$\|\varphi\|_0 := \sum_{k=0}^{r} \sup_{x\in[a,b]}\left|\varphi^{(k)}(x)\right| +$$

$$+ sup\left\{\frac{\left|\varphi^{(r)}(x)-\varphi^{(r)}(\bar{x})\right|}{\gamma(|x-\bar{x}|)};\ x, \bar{x}\epsilon[a, b], x \neq \bar{x}\right\} \tag{5}$$

We will show the equivalence of norms (4) and (5).

Proposition 1.

The norms (4) and (5) are equivalent in $W_\gamma[a, b]$ and the following inequalities hold

$$\|\varphi\| \leq \|\varphi\|_0 \leq K_1\|\varphi\| \tag{6}$$

where

$$K_1 = max\left(\sum_{k=0}^{r+1} d^k\,; \sum_{k=0}^{r+1}(\gamma(d))^k\right) \tag{7}$$

Proof.

The inequality $\|\varphi\| \le \|\varphi\|_0$ is obvious. For $\varphi \in W_\gamma[a,b]$ define the constants

$$M := sup\left\{\frac{|\varphi^{(r)}(x)-\varphi^{(r)}(\bar{x})|}{\gamma(|x-\bar{x}|)};\ x,\bar{x}\epsilon[a,b], x \neq \bar{x}\right\},$$

$$L_k := sup\left\{\frac{|\varphi^{(k)}(x)-\varphi^{(k)}(\bar{x})|}{|x-\bar{x}|};\ x,\bar{x}\epsilon[a,b],\ x \neq \bar{x}\right\}, k = 0,1,\dots,r-1.$$

Since for $\varphi \in W_\gamma[a,b]$ the following inequalities hold

$$\sup_{x\in[a,b]}\left|\varphi^{(r)}(x)\right| \le \left|\varphi^{(r)}(a)\right| + M\gamma(d)$$

$$\sup_{x\in[a,b]}\left|\varphi^{(k)}(x)\right| \le \left|\varphi^{(k)}(a)\right| + L_k d,\ k = 0,1,\dots,r-1,$$

and by the mean value theorem we have

$$L_k \le \sup_{x\in[a,b]}\left|\varphi^{(k+1)}(x)\right|,\quad k = 0,1,\dots,r-1,$$

thus we obtain

$$\sup_{x\in[a,b]}\left|\varphi^{(k)}(x)\right| \le \left|\varphi^{(k)}(a)\right| + \left|\varphi^{(k+1)}(a)\right|d + \dots + \left|\varphi^{(r)}(a)\right|d^{r-k} + Md^{r-k}\gamma(d)$$

In view of above inequalities we have

$$\|\varphi\|_0 \le \sum_{k=0}^{r} \sup_{x\in[a,b]}\left|\varphi^{(k)}(x)\right| +$$
$$+M\sum_{i=0}^{r}\left|\varphi^{(i)}(a)\right|\sum_{k=0}^{i} d^k + M(\gamma(d)\sum_{k=0}^{r} d^k + 1).$$

In the case $\gamma(d) \ge d$ we obtain

$$\gamma(d)\sum_{k=0}^{r} d^k + 1 \le \sum_{k=0}^{r+1}(\gamma(d))^k,$$

then

$$\|\varphi\|_0 \le \sum_{k=0}^{r+1}(\gamma(d))^k\,\|\varphi\|.$$

In the case $\gamma(d) < d$ we have

$$\gamma(d)\sum_{k=0}^{r} d^k + 1 \le \sum_{k=0}^{r+1} d^k,$$

therefore

$$\|\varphi\|_0 \le \sum_{k=0}^{r+1} d^k\,\|\varphi\|.$$

In view of (7) we obtain inequality (6). This completes the proof.

Proposition 2.

If the function f and g belong to $W_\gamma[a,b]$, then their product $f\cdot g$ also belongs to this space and there exists a constant $K_2>0$ such that

$$\|f\cdot g\|_0 \le K_2\|f\|_0 \tag{8}$$

where

$$K_2=\max(P_r,Q_r)$$

$$P_r=\max_{i=0,\dots,r-1}\left(\|g\|_0;\ \sup_{[a,b]}\left|g^{(i)}\right|+2\sup_{[a,b]}\left|g^{(i+1)}\right|\right)$$

$$Q_r=\left(\frac{d}{\gamma(d)}\right)^{r-1}\max_{i=0,\dots,r-1}\left(\left(\frac{d}{\gamma(d)}\right)^{r-1}\|g\|_0;\sup_{[a,b]}\left|g^{(i)}\right|+2\frac{d}{\gamma(d)}\sup_{[a,b]}\left|g^{(i+1)}\right|\right)$$

Proof.

For $f,g\in W_\gamma[a,b]$ we have

$$\left|(fg)^{(r)}(x)-(fg)^{(r)}(\bar{x})\right|=$$

$$=\left|\textstyle\sum_{i=0}^{r}\binom{r}{i}f^{(i)}(x)g^{(r-i)}(x)-\sum_{i=0}^{r}\binom{r}{i}f^{(i)}(\bar{x})g^{(r-i)}(\bar{x})\right|\le$$

$$\le\sup_{x\in[a,b]}|f|\left|g^{(r)}(x)-g^{(r)}(\bar{x})\right|+$$

$$+\textstyle\sum_{i=1}^{r}\binom{r}{i}\displaystyle\sup_{x\in[a,b]}\left|f^{(i)}\right|\left|g^{(r-i)}(x)-g^{(r-i)}(\bar{x})\right|+$$

$$+\textstyle\sum_{i=0}^{r-1}\binom{r}{i}\displaystyle\sup_{x\in[a,b]}\left|g^{(r-i)}\right|\left|f^{(i)}(x)-f^{(i)}(\bar{x})\right|+\sup_{[a,b]}|g|\left|f^{(r)}(x)-f^{(r)}(\bar{x})\right|$$

Let $L_i, K_i,\ i=0,\dots,r-1$, denote Lipschitz constants of the functions $f^{(i)}$ and $g^{(i)}$ respectively, and by M_f, M_g denote γ –Hölder constants of the functions $f^{(r)}$ and $g^{(r)}$ respectively. Put

$$F_i:=\sup_{[a,b]}\left|f^{(i)}\right|,\ i=0,\dots,r,$$

$$G_i:=\sup_{[a,b]}\left|g^{(i)}\right|,\ i=0,\dots,r.$$

Thus by Lemma 1 we get

$$\left|(fg)^{(r)}(x)-(fg)^{(r)}(\bar{x})\right|\le F_0M_g\,\gamma(|x-\bar{x}|)+\textstyle\sum_{i=1}^{r}\binom{r}{i}F_i\,K_{r-i}\,\gamma_0(|x-\bar{x}|)+$$

$$+\textstyle\sum_{i=0}^{r-1}\binom{r}{i}G_{r-i}\,L_i\,\gamma_0(|x-\bar{x}|)+G_0M_f\,\gamma(|x-\bar{x}|),$$

and by definition (3) of function γ_0 there is $f\cdot g\in W_\gamma[a,b]$.

Using mathematical induction we prove the inequality (8).
For $r = 1$ we have

$$\|f \cdot g\|_0 \le \sup_{[a,b]}|f| \sup_{[a,b]}|g| + \sup_{[a,b]}|f| \sup_{[a,b]}|g'| + \sup_{[a,b]}|f'| \sup_{[a,b]}|g| +$$

$$+ \sup \frac{|f'(x)g(x) + f(x)g'(x) - f'(\bar{x})g(\bar{x}) - f(\bar{x})g'(\bar{x})|}{\gamma(|x - \bar{x}|)} \le$$

$$\le \sup_{[a,b]}|f|(\sup_{[a,b]}|g| + \sup_{[a,b]}|g'|) + \sup_{[a,b]}|f'| \sup_{[a,b]}|g| +$$

$$+ \sup_{[a,b]}|f| \, sup \frac{|g'(x) - g'(\bar{x})|}{\gamma(|x - \bar{x}|)} + \sup_{[a,b]}|g| \, sup \frac{|f'(x) - f'(\bar{x})|}{\gamma(|x - \bar{x}|)} +$$

$$+ \sup_{[a,b]}|f'| \, sup \frac{|g(x) - g(\bar{x})|}{\gamma(|x - \bar{x}|)} + \sup_{[a,b]}|g'| \, sup \frac{|f(x) - f(\bar{x})|}{\gamma(|x - \bar{x}|)}$$

where supremes of the above rational expressions are for $x, \bar{x} \epsilon [a, b], x \neq \bar{x}$. (In the next inequalities in this proof the same notations are used).

If $\gamma(d) \ge d$ then the following inequalities hold the function g:

$$sup \frac{|g(x) - g(\bar{x})|}{\gamma(|x - \bar{x}|)} \le sup \frac{|g(x) - g(\bar{x})|}{|x - \bar{x}|} \le \sup_{[a,b]}|g'|$$

and the same for function f. From the above we obtain

$$\|f \cdot g\|_0 \le \sup_{[a,b]}|f| \left((\sup_{[a,b]}|g| + \sup_{[a,b]}|g'| + sup \frac{|g'(x) - g'(\bar{x})|}{\gamma(|x - \bar{x}|)} \right) +$$

$$+ \sup_{[a,b]}|f'|(\sup_{[a,b]}|g| + 2\sup_{[a,b]}|g'|) + sup \frac{|f'(x) - f'(\bar{x})|}{\gamma(|x - \bar{x}|)} \sup_{[a,b]}|g| \le$$

$$\le \|f\|_0 max \left(\|g\|_0; \ \sup_{[a,b]}|g| + 2\sup_{[a,b]}|g'| \right) = P_1 \|f\|_0$$

If $\gamma(d) < d$, by (3) we have the following inequalities

$$sup \frac{|g(x) - g(\bar{x})|}{\gamma_0(|x - \bar{x}|)} \le sup \frac{|g(x) - g(\bar{x})|}{|x - \bar{x}|} \le \sup_{[a,b]}|g'|$$

and using definition (2) we obtain

$$sup \frac{|g(x) - g(\bar{x})|}{\gamma(|x - \bar{x}|)} \le \frac{d}{\gamma(d)} \sup_{[a,b]}|g'|$$

(similarly for the function f).

Therefore we have

$$\|f\cdot g\|_0 \leq \sup_{[a,b]}|f|\left(\leq \sup_{[a,b]}|f|(\sup_{[a,b]}|g| + \sup_{[a,b]}|g'| + sup\frac{|g'(x)-g'(\bar{x})|}{\gamma(|x-\bar{x}|)}\right) +$$

$$+\sup_{[a,b]}|f'|(\sup_{[a,b]}|g| + 2\frac{d}{\gamma(d)}\sup_{[a,b]}|g'|) + sup\frac{|f'(x)-f'(\bar{x})|}{\gamma(|x-\bar{x}|)}\sup_{[a,b]}|g| \leq$$

$$\leq \|f\|_0 max\left(\|g\|_0;\sup_{[a,b]}|g| + 2\frac{d}{\gamma(d)}\sup_{[a,b]}|g'|\right) = Q_1\|f\|_0$$

Putting

$$K_2 = \max(P_1, Q_1)$$

we get the inequality (8) for $r = 1$.
Let functions f, g have $(k+1)$-order derivatives. Denote by $\|\cdot\|_{0,k}$ the norm $\|\cdot\|_0$ for the functions k times differentiable. Suppose that the inequality (8) is true for $k \geq 1, k \in N$, i.e.

$$\|f\cdot g\|_{0,k} \leq K_2\|f\|_{0,k}$$

where

$$K_2 = \max(P_k, Q_k)$$

$$P_k = \max_{i=0,\ldots,k-1}\left(\|g\|_{0,k};\ \sup_{[a,b]}\left|g^{(i)}\right| + 2\sup_{[a,b]}\left|g^{(i+1)}\right|\right)$$

$$Q_k = \left(\frac{d}{\gamma(d)}\right)^{k-1}\max_{i=0,\ldots,k-1}\left(\left(\frac{d}{\gamma(d)}\right)^{k-1}\|g\|_{0,k};\ \sup_{[a,b]}\left|g^{(i)}\right| + 2\frac{d}{\gamma(d)}\sup_{[a,b]}\left|g^{(i+1)}\right|\right)$$

We prove that the inequality (8) holds for $(k+1)$. We have

$$\|f\cdot g\|_{0,k+1} \leq \sup_{[a,b]}|f|\sup_{[a,b]}|g| + \|f\cdot g'\|_{0,k} + \|f'\cdot g\|_{0,k}$$

If $\gamma(d) \geq d$ then the following inequality holds

$$\|f\cdot g\|_{0,k+1} \leq \sup_{[a,b]}|f|\sup_{[a,b]}|g| +$$

$$+\|f\|_{0,k}\max_{i=0,\ldots,k-1}\left(\|g'\|_{0,k};\ \sup_{[a,b]}\left|g^{(i+1)}\right| + 2\sup_{[a,b]}\left|g^{(i+2)}\right|\right) +$$

$$+\|f'\|_{0,k}\max_{i=0,\ldots,k-1}\left(\|g\|_{0,k};\ \sup_{[a,b]}\left|g^{(i)}\right| + 2\sup_{[a,b]}\left|g^{(i+1)}\right|\right)$$

Since

$$\|f\|_{0,k} \le \|f\|_{0,k+1} \; ; \; \|f'\|_{0,k} \le \|f'\|_{0,k+1}$$

(the same inequalities hold for the function g) thus we have

$$\|f\cdot g\|_{0,k+1} \le \|f\|_{0,k+1} \max_{i=0,\dots,k}\left(\|g\|_{0,k+1};\ \sup_{[a,b]}\left|g^{(i)}\right| + 2\sup_{[a,b]}\left|g^{(i+1)}\right|\right) =$$
$$= P_{k+1}\|f\|_{0,k+1}$$

In the case $\gamma(d) < d$ we have

$$\|f\|_{0,k} \le \frac{d}{\gamma(d)}\|f\|_{0,k+1};\ \|f'\|_{0,k} \le \|f'\|_{0,k+1}$$

(the same inequalities hold for the function g). Therefore

$$\|f\cdot g\|_{0,k+1} \le \sup_{[a,b]}|f| \sup_{[a,b]}|g| +$$
$$+\|f\|_{0,k}\left(\frac{d}{\gamma(d)}\right)^{k} \max_{i=0,\dots,k-1}\left(\left(\frac{d}{\gamma(d)}\right)^{k-1}\|g'\|_{0,k};\ \sup_{[a,b]}\left|g^{(i+1)}\right|\right.$$
$$\left. + 2\frac{d}{\gamma(d)}\sup_{[a,b]}\left|g^{(i+2)}\right|\right) +$$
$$+\|f'\|_{0,k}\left(\frac{d}{\gamma(d)}\right)^{k-1} \max_{i=0,\dots,k-1}\left(\left(\frac{d}{\gamma(d)}\right)^{k-1}\|g\|_{0,k};\ \sup_{[a,b]}\left|g^{(i)}\right|\right.$$
$$\left. + 2\frac{d}{\gamma(d)}\sup_{[a,b]}\left|g^{(i+1)}\right|\right) \le$$

$$\le \|f\|_{0,k+1}\left(\frac{d}{\gamma(d)}\right)^{k} \max_{i=0,\dots,k-1}\left(\left(\frac{d}{\gamma(d)}\right)^{k}\|g\|_{0,k+1};\ \sup_{[a,b]}\left|g^{(i)}\right| +\right.$$
$$\left. + 2\frac{d}{\gamma(d)}\sup_{[a,b]}\left|g^{(i+1)}\right|\right) = Q_{k+1}\|f\|_{0,k+1}$$

Putting $K_2 = max(P_{k+1}, Q_{k+1})$ we get the inequality (8) for $(k+1)$. This completes the proof.

Conclusions

In this paper some properties of the Banach space $W_\gamma[a,b]$ have been presented. They will be applied in the forthcoming papers.

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