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SEMIPRIME FDI-RINGS

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Abstract. In this paper we present some results for FDI-rings, i.e. rings with a complete set of pairwise orthogonal primitive idempotents. We consider the nilpotency index of ideals and give its upper band for ideals in some classes of rings. We also give a new proof of a criterion of semiprime FDI-rings to be prime.

Keywords: *semiprime rings, prime rings, nilpotent ideal, nilpotency index, Jacobson radical, semiperfect ring, complete set of idempotents, FDI-rings*

Introduction

In this paper we consider FDI-rings, i.e. rings with a finite decomposition of identity into a sum of pairwise orthogonal primitive idempotents, which form the important class of rings with finiteness conditions. We present some results for these rings.

In section 1 we consider a criterium of nilpotency for ideals and give some upper bounds for nilpotency index of nilpotent ideals in some classes in rings.

Some properties of FDI-rings connected with idempotents in them are considered in section 2.

In section 3 we present a new proof of the theorem which yields a criterion for a semiprime FDI-ring to be decomposable into a finite direct product of prime rings. This theorem was first proved in [1] and it can be considered as a generalization of the theorem for semiprime semiperfect rings proved in [2].

The proof of the theorem for semiprime FDI-rings given here in this paper is much more similar to the proofs of structure theorems for semihereditary FDI-rings given by Drozd in [3] and for piecewise domains given by Gordon and Small in [4]. Thus, it shows the close connections between these classes of rings.

Throughout this paper all rings are associative with identity.

1. The nilpotency index of ideals

The notion of nilpotency is very important in the theory of rings and algebras, just as it is in other different fields of mathematics. Recall that a non-zero element

a of a ring A is called **nilpotent** if there exists an integer $n > 0$ such that $a^n = 0$. The smallest such n is called the **nilpotency index** of a. A right (or left) ideal is called a **nil-ideal** if its every element is nilpotent. The set $N(A)$ of all nilpotent elements of A is its two-sided ideal and it is called the **nilradical** of A. A right ideal I of a ring A is called **nilpotent** if $I^n = 0$ for some positive integer n. The smallest integer $n > 0$ such that $I^n = 0$ but $I^{n-1} \neq 0$ is called the **nilpotency index** (or **index of nilpotency**) of the ideal I and we write it $t(I)$. It is well known that the Jacobson radical of a ring A contains all one-sided nil-ideals, the prime radical of a ring A is a nil-ideal and it contains all nilpotent one-sided ideals of A. If a ring A is right Noetherian, then every one-sided nil-ideal of A is nilpotent, by the theorem of Levitzki (see [5]), and the prime radical of A is the largest nilpotent ideal in A (see [6, Proposition 11.2.11]). Also there are well-known examples of rings containing nil-ideals that are not nilpotent.

The sum of all nil-ideals in A is its two-sided ideal $N(A)$ and it is called the **nilradical** of R. It is well known that $N(A)$ is the largest nil-ideal in A and there are rings for which the nilradical is not nilpotent.

Recall that an element e of a ring A is called an **idempotent** if $e^2 = e$. If $e \in A$ is an idempotent then e and $1 - e$ are orthogonal idempotents such that $1 = e + (1 - e)$. Obviously, eAe is a ring with identity e for any idempotent $e \in A$. If I is a two-sided ideal in A, then eIe is a two-sided ideal in a ring eAe.

In this section we will give some corollaries from the following important statement whose proof can be found in [5]:

Proposition 1.1. [5, Lemma 2.7.13]. Suppose e is an idempotent of a ring A and I is a two-sided ideal of A. Then I is nilpotent if and only if eIe and $(1-e)I(1-e)$ are nilpotent.

The proof of this proposition can be found in [5], but we include it for the sake of completeness and to obtain important corollaries using this proof.

Proof.

1. Let I be a two-sided nilpotent ideal of A with nilpotency index $t(I) = n$. Suppose that $e^2 = e$ is an idempotent of A and $x_1, x_2, \ldots, x_n \in eIe$. Then $x_i = ey_ie$ for some $y_i \in I$, $i = 1, 2, \ldots, n$, and $x_1x_2\ldots x_n = ey_1e\, ey_2e\ldots ey_ne = ey_1ey_2e\ldots y_ne = z_1z_2\ldots z_n = 0$, where $z_i = ey_i \in eI$.
2. For the converse, suppose that $e^2 = e$ is an idempotent of A and ideals eIe and $(1-e)I(1-e)$ are nilpotent with $t(eIe) = n$ and $t((1 - e)I(1 - e)) = m$. Put $k = m + n$. Let $x_1, x_2, \ldots, x_n \in I$. Since $1 = e + (1 - e)$,

$$u = x_1x_2\ldots x_k = [e + (1 - e)]x_1[e + (1 - e)]x_2[e+(1-e)]\ldots [e+(1-e)]x_k[e+(1-e)] =$$
$$= \sum e_{i_1} x_1 e_{i_2} x_2 \ldots e_{i_k} x_k e_{i_{k+1}},$$

where each idempotent e_{i_s} is either e or $1 - e$. Consider an element $y = e_{i_1}x_1e_{i_2}x_2...e_{i_k}x_ke_{i_{k+1}}$. Since we have $k + 1 > n + m$ idempotents in this product, either the number of idempotent e is more than n or the number of idempotent $1 - e$ is more than m in this product. In any case, each $y = 0$, and so $u = 0$. Therefore I is a nilpotent ideal.

Right from the proof of Proposition 1.1 we have the following result.

Corollary 1.2. Suppose e is an idempotent of a ring A and I is a two-sided ideal of A with nilpotency index $t(I)$. Then

$$t(I) \le n + m \tag{1}$$

where $n = t(eIe)$ and $m = t[(1-e)I(1-e)]$.

Definition 1.3. A finite set of pairwise orthogonal idempotents $e_1, e_2,..., e_m$ of a ring A is called **complete** if

$$e_1 + e_2 + \ldots + e_m = 1 \in A. \tag{2}$$

Theorem 1.4. Let A be a ring with finite complete set of pairwise orthogonal idempotents $S = \{e_1 , e_2 ,\ldots,e_k\}$. Then a two-sided ideal I of A is nilpotent if and only if eIe is nilpotent for any idempotent $e \in S$. Moreover,

$$t(I) \le n_1 + n_2 + ... + n_k. \tag{3}$$

where $n_i = t(e_iIe_i)$.

Proof.
Apply the induction to Proposition 1.1 and Corollary 1.2.

By Proposition 1.1, the theorem is valid if $k = 2$ in the decomposition (2). Assume that theorem is true if $n \le k - 1$. Set $e = e_1 + e_2 + ... + e_{k-1}$, $f = 1 - e = e_k$. Then eIe and fIf are nilpotent, by Proposition 1.1, and $t(I) \le m_{k-1} + n_k$, where $m_{k-1} = t(eIe)$ and $n_k = t(fIf)$. Since eIe is a nilpotent two-sided ideal in eAe, by assumption

$$t(eIe) \le n_1 + n_2 + ... + n_{k-1}.$$

So, $t(I) \le n_1 + n_2 + ... + n_k$.

Recall that A is a semiperfect ring if any finitely generated A-module has a projective cover [7]. This is equivalent to the condition that the identity of A can be decomposed into a finite sum of pairwise orthogonal local idempotents [8]. So that a ring A is semiperfect if it has a finite complete set of local idempotents. If e is a non-zero idempotent of a semiperfect ring A then eAe is also a semiperfect ring with the identity e (see [6, Corollary 10.3.11]). Applying Theorem 2.4 to a semiperfect ring we obtain the following result which is some extension of [6, Theorem 11.4.1].

Proposition 1.5. Suppose A is a semiperfect ring with a finite complete set of pairwise orthogonal local idempotents $S = \{e_1, e_2, \ldots, e_m\}$. Then a two-sided ideal I in A is nilpotent if and only if eIe is nilpotent for any idempotent $e \in S$. Moreover,

$$t(I) \leq n_1 + n_2 + \ldots + n_k. \tag{4}$$

where $n_i = t(e_iIe_i)$.

In particular, if $e_iIe_i = 0$ for every local idempotent $e_i \in A$ from the decomposition (4), then I is nilpotent.

In particular, if $I = J(A)$ is the Jacobson radical of a semiperfect ring A, we immediately obtain the following corollary.

Corollary 1.6. The Jacobson radical $J(A)$ of a semiperfect ring A is nilpotent if an ideal $e_iJ(A)e_i$ is nilpotent for every local idempotent $e_i \in A$ in some decomposition (4) of the identity of A into a finite sum of pairwise orthogonal local idempotents. In particular, if $e_iJ(A)e_i = 0$ for every local idempotent $e_i \in A$ from some decomposition (4), then $J(A)$ is nilpotent.

2. FDI-rings

Recall that an idempotent $e^2 = e \in A$ is called **primitive** if it cannot be written as a sum of two non-zero pairwise orthogonal idempotents of A.

Consider the following important class of rings with finiteness condition.

Definition 2.1. [6, Chapter 2]. A ring A is called an **FDI-ring** if it has a (finite) complete set $S = \{e_1, e_2, \ldots, e_n\}$ of orthogonal primitive idempotents, i.e. there exists a decomposition of the identity $1 \in A$ into a finite sum $1 = e_1 + \ldots + e_n$ of pairwise orthogonal primitive idempotents.

In this case the right (left) regular A-module A_A (${}_AA$) can be decomposed into a finite direct sum of indecomposable modules e_iA (Ae_i).

Note that the decomposition of $1 \in A$, given in the definition of an FDI-ring, may not be unique. Sometimes FDI-rings called rings having enough finite set of idempotents (see [4]).

Examples 2.2. The following rings are FDI-rings:

1. Division rings.
2. Finite direct sums of FDI-rings.

3. Rings which are finite dimensional vector spaces over a division ring.
4. The ring of matrices $M_n(D)$ over a division ring D.
5. Semisimple rings.
6. Right (left) Artinian rings.
7. Right (left) Noetherian rings.
8. Semiperfect rings.
9. Right finite-dimensional rings, i.e rings which do not contain infinite direct sum of non-zero right ideals.
10. Perfect rings.

Recall the following important results:

Proposition 2.3. (See [5, Theorem 2.4.14, Corollary 2.4.15]).
(i) Let A be an FDI-ring. Then the identity of A can be written as a sum of a finite number of orthogonal centrally primitive idempotents.
(ii) Any FDI-ring can be uniquely decomposed into a direct product of a finite number of indecomposable rings.

Lemma 2.4. Let A be an FDI-ring with finite complete set $S = \{e_1, e_2, \ldots, e_n\}$ of orthogonal primitive idempotents. Then cAc is also an FDI-ring for any idempotent $c = c_1 + c_2 + \ldots + c_m$, where each $c_i \in P \subseteq S$.

Proof.
Let A be an FDI-ring with a complete set of primitive pairwise orthogonal idempotents $S = \{e_1, e_2, \ldots, e_n\}$, i.e. there is a decomposition $1 = e_1 + e_2 + \ldots + e_n$ of the identity $1 \in A$ into a sum of pairwise orthogonal primitive idempotents. Let $c = c_1 + c_2 + \ldots + c_m$, where each $c_i \in P \subseteq S$, be a non-zero idempotent of A.

We will show that if $ce_ic \neq 0$ then ce_ic is a primitive idempotent in cAc. To this end it suffices to show that a ring $B = (ce_ic)(cAc)(ce_ic)$ has the only idempotent ce_ic. If e_i does not belong to P then $ce_ic = 0$. If $e_i \in P$ then $ce_ic = e_ic = ce_i = e_i$. So that $B = (ce_ic)(cAc)(ce_ic) = e_iAe_i$ and $ce_ic = e_i$ is the only idempotent in B.

Now taking into account this lemma, we can obtain the following corollary from Theorem 1.4.

Proposition 2.5. Let A be an FDI-ring with finite complete set of pairwise orthogonal primitive idempotents $S = \{e_1, e_2, \ldots, e_m\}$. Then a two-sided ideal I of A is nilpotent if and only if eIe is nilpotent for any idempotent $e \in S$. Moreover,

$$t(I) \leq n_1 + n_2 + \ldots + n_k. \quad (5)$$

where $n_i = t(e_iIe_i)$.

In particular, if $e_iIe_i = 0$ for every primitive idempotent $e_i \in S$, then I is nilpotent.

Definition 2.6. An idempotent $e \in A$ is called **central** if $ea = ae$ for all $a \in A$.

It is easy to show that an idempotent $e \in A$ is central if and only if

$$eA(1-e) = (1-e)Ae = 0,$$

if and only if

$$eA = Ae = eAe,$$

and so that if and only if a two-sided decomposition of A has the following form:

$$A = \begin{bmatrix} eAe & 0 \\ 0 & (1-e)A(1-e) \end{bmatrix}.$$

Definition 2.7. A central idempotent $c \in A$ is called **centrally primitive** if it cannot be written as a sum of two non-zero pairwise orthogonal central idempotents of A.

Some generalizations of a notion of a central idempotent are left (or right) semicentral idempotents introduced by Birkenmeier et al. in [9].

Definition 2.8. [9]. An idempotent $e^2 = e \in A$ is called **left** (resp. **right**) **semicentral** in A if $Ae = eAe$ (resp. $eA = eAe$).

Obviously, any central idempotent is left and right semicentral.

An idempotent $e \in A$ is left semicentral if and only if

$$(1-e)Ae = (1-e)eAe = 0,$$

and if and only if a two-sided decomposition of A has the following form:

$$A = \begin{bmatrix} eAe & eA(1-e) \\ 0 & (1-e)A(1-e) \end{bmatrix}.$$

An idempotent $e \in A$ is left semicentral if and only if for each $x \in A$:

$$xe = [e + (1-e)]exe = exe + (1-e)xe = exe.$$

So that we obtain the following lemma.

Lemma 2.9. (See [9, Lemma 1.1]) The following conditions are equivalent for an idempotent $e^2 = e \in A$:
(1) $Ae = eAe$.
(2) $(1-e)Ae = 0$.

(3) $xe = exe$ for each $x \in A$.
(4) A two-sided Peirce decomposition of A has the following form:

$$A = \begin{bmatrix} eAe & eA(1-e) \\ 0 & (1-e)A(1-e) \end{bmatrix}$$

Similar conditions hold for right semicentral idempotents which we formulate in the following lemma.

Lemma 2.10. (See [9, Lemma 1.1]) The following conditions are equivalent for an idempotent $e^2 = e \in A$:
(1) $eA = eAe$.
(2) $eA(1-e) = 0$.
(3) $ex = exe$ for each $x \in A$.
(4) A two-sided Peirce decomposition of A has the following form:

$$A = \begin{bmatrix} eAe & 0 \\ (1-e)Ae & (1-e)A(1-e) \end{bmatrix}$$

We will use the following result which was proved in [9].

Lemma 2.11. (See [9, Lemma 2.13].) Let $c = c^2$ be a non-zero left (or right) semicentral idempotent of an FDI-ring A. Then cAc is also an FDI-ring. Moreover, if $\mathrm{S} = \{e_1, e_2, \ldots, e_n\}$ is a complete set of primitive idempotents of A then there is a subset $\mathrm{P} \subseteq \mathrm{S}$ such that

$$\{cg_ic \mid g_i \in \mathrm{P}\}$$

is a complete set of primitive idempotents of cAc. If $c \neq 1$ then the set P has less than n elements.

From this lemma and Proposition 2.4 we immediately obtain the following corollary.

Corollary 2.12.
1. Let A, B be rings and X be an A-B-bimodule. Then the ring

$$M = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$$

   is an FDI-ring if and only if A and B are both FDI-rings.
2. Any FDI-ring can be uniquely decomposed into a direct product of a finite number of indecomposable FDI-rings.

3. Semiprime FDI-rings

In this section we prove a criterion for a semiprime FDI-ring to be decomposable into a finite direct product of prime rings. This theorem was considered in [7]. Here we give a new proof of this theorem without use of induction on the number of idempotents.

Recall that ring A is **prime** if the product of any two non-zero two-sided ideals I, J in A is not equal to zero, i.e. $IJ = 0$ implies that $I = 0$ or $J = 0$. A ring A is **semiprime** if it does not contain non-zero nilpotent ideals, i.e. $I^n = 0$ for two-sided non-zero ideal I in A implies that $I = 0$.

The easiest example of a prime commutative ring is a ring $\mathbf{Z}$ of integers and the ring $\mathbf{Z}\oplus\mathbf{Z}$ is a semiprime commutative ring. The ring

$$\begin{bmatrix} \mathbf{Z} & n\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix}$$

where n is a positive integer, is a prime noncommutative FDI-ring.

The ring

$$\begin{bmatrix} \mathbf{Z} & n\mathbf{Z} \\ 0 & \mathbf{Z} \end{bmatrix}$$

is an FDI-ring which is not prime.

Recall the following important result about prime and semiprime rings.

Proposition 3.1. (See [4, Proposition 9.2.13]) If $e^2 = e \in A$ is a non-trivial idempotent of prime (semiprime) ring A then the ring eAe is also prime (semiprime).

We will also use the following lemma which was proved in [1].

Lemma 3.2. (See [1, Lemma 3.3]) Let A be a semiprime ring and $1 = g_1 + g_2$ be decomposition of $1 \in A$ into a sum of two pairwise orthogonal idempotents, $A = \begin{bmatrix} A_1 & X \\ Y & A_2 \end{bmatrix}$, where $A_1 = g_1 A g_1$, $A_2 = g_2 A g_2$, $X = g_1 A g_2$, and $Y = g_2 A g_2$. Let $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ be an ideal in A and $M_{12} \neq 0$. Then $M_{12}M_{21} \neq 0$, $M_{21} \neq 0$, $M_{21}M_{12} \neq 0$. Symmetrically, if $M_{21} \neq 0$, then $M_{12} \neq 0$, $M_{12}M_{21} \neq 0$, $M_{21}M_{12} \neq 0$. In particular, if A is an indecomposable ring and $Y \neq 0$, then $YX \neq 0$, $X \neq 0$, $XY \neq 0$.

Lemma 3.3. Let A be a semiprime ring and $1 = g_1 + g_2$ be a decomposition of $1 \in A$ into a sum of two pairwise orthogonal idempotents, $A = \begin{bmatrix} A_1 & X \\ Y & A_2 \end{bmatrix}$, where $A_1 = g_1 A g_1$, $A_2 = g_2 A g_2$, $X = g_1 A g_2 \neq 0$, and $Y = g_2 A g_2 \neq 0$.
If $I = \begin{bmatrix} I_1 & I_{12} \\ I_{21} & I_2 \end{bmatrix}$ is a two-sided ideal in A with $I_{12} \neq 0$ then $I_1 \neq 0$ and $I_2 \neq 0$.

Proof.
If $I_{12} \neq 0$ then $I_{21} \neq 0$ as well by Lemma 3.2. Suppose that $I_1 = 0$ then $I = \begin{bmatrix} 0 & I_{12} \\ I_{21} & I_2 \end{bmatrix}$.
Since I is a two-sided ideal,

$$\boldsymbol{I^2} = \begin{bmatrix} I_{12}I_{21} & I_{12}I_2 \\ I_2 I_{21} & I_{21}I_{12} + I_2^{\ 2} \end{bmatrix} \subseteq \begin{bmatrix} 0 & I_{12} \\ I_{21} & I_2 \end{bmatrix}.$$

Therefore $I_{12}I_{21} = 0$ which implies $I_{12} = I_{21} = 0$ by Lemma 3.2. This contradiction shows that $I_1 \neq 0$. Analogously one can show that $I_2 \neq 0$.

Definition 3.4. An indecomposable projective right A-module P of an FDI-ring A will be called principal if $P \cong e_i A$ for some primitive idempotent e_i, $i = 1,...,n$.

Theorem 3.5. Let A be a semiprime FDI-ring. Suppose that $1 \in A$ has the following decomposition into a finite sum of pairwise orthogonal primitive idempotents:

$$1 = e_1 + e_2 + \ldots + e_n.$$

Then A is a finite direct product of prime rings if and only if all rings $e_i A e_i$ are prime for all $i = 1,2,...,n$.

Proof.
Let A be a semiprime ring, and let $1 = e_1 + e_2 + \ldots + e_n$ be a decomposition of the identity of A into a sum of pairwise orthogonal primitive idempotents. First suppose that A is a finite direct product of prime rings. Say

$$A = A_1 \times A_2 \times \ldots \times A_s.$$

Let e be a non-trivial primitive idempotent of A. Then $eAe \subset A_i$ for some (unique) i. Now $\varphi : I \to eIe$ is a product preserving a surjective map from ideals in A_i to ideals in $eAe \subset A_i$. Indeed let J be an ideal of eAe. Then $A_i J A_i$ is an ideal in A_i and $eA_i J A_i e = J$. Now let $J, J' \subset eAe$ be two non-zero ideals such that $JJ' = 0$. Then $A_i J A_i$ and $A_i I' A_i$ are two ideals in A_i with product zero. So at least one of them is zero as A_i is prime. So at least one of J, J' is zero. This proves that eAe is prime.

We will now prove the converse statement. Let A be a semiprime FDI-ring with decomposition $1 = e_1 + e_2 + \ldots + e_n$ of $1 \in A$ into a sum of primitive pairwise orthogonal idempotents.

Write $e_iAe_j = A_{ij}$ and $P_i = e_iA$. It follows from Lemma 3.2 that either $A_{ij} = A_{ji} = 0$ or $A_{ij} \neq 0$ and $A_{ji} \neq 0$. This yields the possibility to introduce a relation $\approx$ on the set $\{1,2,...,n\}$, setting $i \approx j$ if and only if $A_{ij} \neq 0$. In this case we write $e_iA \approx e_jA$ and the principal projective modules P_i and P_j will be called **equivalent**. We now show that $\approx$ is an equivalence relation on $\{1,2,...,n\}$. Indeed, $i \approx i$ is obvious. Let $i \approx j$, i.e. $A_{ij} \neq 0$. Then from Lemma 3.2 it follows that $A_{ji} \neq 0$, i.e. $j \approx i$. Suppose now that $i \approx j$ and $j \approx k$. Then A_{ij}, A_{ji}, A_{jk}, and A_{kj} are all non-zero. Suppose that i is not equivalent to k, i.e. $A_{ik} = 0$. By Lemma 3.2 $A_{ki} = 0$ as well. Suppose $f = e_i + e_j + e_k$ and consider a ring fAf which has the following two-sided Peirce decomposition:

$$fAf = \begin{bmatrix} A_{ii} & A_{ij} & 0 \\ A_{ji} & A_{jj} & A_{jk} \\ 0 & A_{kj} & A_{kk} \end{bmatrix}.$$

This implies that $A_{ij}A_{jk} = 0$. Therefore $IJ = (A_{ji}A_{ij})(A_{jk}A_{kj}) = 0$, where $I = A_{ji}A_{ij}$ and $J = A_{jk}A_{kj}$ are non-zero ideals in A_j. Since A_j is a prime ring, $I = 0$ or $J = 0$. This contradiction shows that $i \approx k$. So that $\approx$ is an equivalence relation on $\{1,2,...,n\}$.

Let $C_1, C_2, ..., C_m$ be equivalence classes on $\{1,2,...,n\}$ with regard to the relation $\approx$. Set $f_i = \sum_{j \in C_i} e_j$, then the $f_1, f_2, ..., f_m$ form a complete set of orthogonal idempotents in A.

We now show that A is isomorphic to a direct product of rings: $A \cong \prod_{i=1}^{m} f_iAf_i$.

Indeed, let $k \in C_i$ and $s \in C_j$ for $i \neq j$. This means that k is not equivalent to s and so $A_{ks} = A_{sk} = 0$, i.e. $f_iAf_j = f_jAf_i = 0$ for all $i \neq j$. Hence $A \cong \prod_{i=1}^{m} f_iAf_i$. Moreover, if $B = f_iAf_i$ then B_B is equal to a direct sum of equivalent principal modules $P_k = e_kA$ where $k \in C_i$.

So without loss of generality, by corollary 3.10(2), one can assume that A is an indecomposable semiprime FDI-ring with decomposition $1 = e_1 + e_2 + \ldots + e_n$ of $1 \in A$ into a sum of primitive pairwise orthogonal idempotents and all principal right A-modules $P_i = e_iA$ are equivalent. This means that $A_{jk} = e_jAe_k \neq 0$ for all $j, k = 1,2,...,n$.

Suppose that all the A_{ii} are prime rings. We will show that A is a prime ring. Let I, J be a non-zero two-sided ideal in A such that $IJ = 0$. Since I,J are non-zero ideals there are indices i, j, k, s such that $I_{ij} \neq 0$ and $J_{ks} \neq 0$. Let $g = e_i + e_j$ and $f = e_k + e_s$. Then gIg and fJf are two-sided ideals in semiprime rings gAg and fAf respectively, by Proposition 3.1. Therefore by Lemma 4.2 $I_i \neq 0$ and $J_k \neq 0$. Consider $M = I_i A_{ik} J_k \subseteq IJ$ where $A_{ik} \neq 0$ by assumption. We will show that $M \neq 0$.

Suppose that $L = A_{ik} J_k = 0$. Then $A_{ki} A_{ik} J_k = 0$. Since $A_{ki} \neq 0$, $S_k = A_{ki} A_{ik} \neq 0$ by Lemma 3.2. So that S_k is a non-zero two-sided ideal in A_k and $S_k J_k = 0$ in a prime ring A_k with $S_k \neq 0$ and $J_k \neq 0$. This contradiction shows that $L \neq 0$. Suppose now that $L A_{ki} = A_{ik} J_k A_{ki} = 0$. Then $A_{ki} A_{ik} J_k A_{ki} A_{ik} = S_k J_k S_k = 0$ in a prime ring A_k. So that $V_i = A_{ik} J_k A_{ki} \neq 0$ and V_i is a two-sided ideal in A_i. Since $M = I_i A_{ik} J_k = 0$ implies $M A_{ki} = I_i A_{ik} J_k A_{ki} = I_i V_i = 0$ in a prime ring A_i with $I_i \neq 0$ and $V_i \neq 0$, one obtains that $M = I_i A_{ik} J_k \neq 0$. Thus $IJ \neq 0$, i.e. A is a prime ring.

Conclusion

In this paper we present some results about FDI-rings, which are rings with a finite complete set of primitive orthogonal idempotents. In particular, it is given a new proof of a criterion for semiprime FDI-rings to be prime. We also consider the nilpotency index of ideals and give the upper bound of the nilpotency index, in particular for ideals in semiperfect and FDI-rings.

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