WEAK CONVERGENCE IN L^1 OF THE SEQUENCES OF MONOTONIC FUNCTIONS

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Abstract. We use (quasi-) Young measures associated with strictly monotonic functions with a differentiable inverse to prove an $L^1([0,1],\mathbb{R})$ weak convergence of the monotonic sequence of such functions. The result is well known, but the method seems to be new.

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Introduction

Weak convergence of the sequences in a normed vector space is one of the crucial concepts of applied functional analysis. It is also a classical one. In any functional analysis textbook we can find theorems characterizing weak convergence in concrete normed spaces. In this paper we would like to use a tool somewhat connected with the weak convergence, but used in a different context so far: the Young measures.

Young measures are the abstract tool discovered by Laurence Chisolm Young while examining functionals with nonconvex integrands in the calculus of variations (see [1]). Such functionals (for instance energy functionals in the nonlinear elasticity) usually do not attain their infima. The elements of their minimizing sequences, which are not convergent in norm, but merely weakly in an appropriate function space, oscillate more and more rapidly around their weak limits. Young measures, in a sense, capture some information about these oscillations.

Explicit calculation of the Young measures associated with weakly convergent sequences of oscillating functions is not an easy task. However, we can often take simpler objects under consideration, the so-called *quasi-Young measures*. Their explicit form can be obtained much simpler. Moreover, in many cases they appear to be exactly the Young measures (see [2]). This is connected with the fact that we can look at the Young measures not only as an object associated with a sequence. It turns out that we can associate a Young measure with any measurable function whose range lies in a compact set.

A particularly simple case holds if the considered function is defined on an interval with values in $\mathbb{R}$ and is strictly monotonic with a continuously differentiable inverse. In this case the associated Young measure is a Lebesgue-Stieltjes measure. This in connection with the fact that Young measure is a probability measure makes it possible to use the Young measures in establishing the weak convergence results.

These results are of course not new and are limited to a rather special case, but nevertheless it seems to be a nice application of abstract objects used in quite different areas so far.

1. Preliminaries

In this section we recall definitions and theorems needed in the sequel.

1.1. Weak convergence in the space of integrable functions

Definition 1.1.

a. Let $(X,\mathcal{F},\mu)$ be a measure space. Let $\mathcal{L}^1_\mu(X)$ be the set of all μ-measurable functions $f: X \to \mathbb{R}$ such that

$$\int_X |f| d\mu < \infty.$$

By $L^1(X)$ we will denote the set of all equivalence classes of functions from $\mathcal{L}^1_\mu(X)$ equal to μ - a.e. We will call $L^1(X)$ a space of functions integrable on X with respect to the measure μ (shortly: the space of integrable functions, if the space X and the measure μ are fixed).

b. Let $(X,\mathcal{F},\mu)$ be a measure space and let $g : X \to \mathbb{R}$ be a measurable function. We say that g is essentially bounded on X if the number

$$\inf\{M > 0: |g(x)| \le M \; \mu - a.e. \text{ in } X\},$$

called the essential supremum of g is finite. The set of all such functions is denoted by $\mathcal{L}^\infty_\mu(X)$. By $L^\infty(X)$ we will denote the set of all equivalence classes of functions from $\mathcal{L}^\infty_\mu(X)$ equal to μ - a.e. We will call $L^\infty(X)$ a space of (essentially) bounded functions on X (shortly: the space of bounded functions).

The space $L^1(X)$ endowed with a norm

$$\|f\| := \int_X |f| d\mu$$

is a Banach space. Further, the space $L^\infty(X)$ endowed with a norm

$$\|g\|_\infty := \inf\{M > 0: |g(x)| \le M \; \mu - a.e. \text{ in } X\}$$

is also a Banach space.

The space of bounded linear functionals on $L^1(X)$, i.e. the space $\left(L^1(X)\right)^*$, is equal to $L^\infty(X)$.

We now recall the notion of a weak convergence in the space $L^1(X)$.

Definition 1.2. Let $(f_n), n \in \mathbb{N}$, be a sequence of functions in $L^1(X)$ and let $f \in L^1(X)$. We say that (f_n) converges weakly to f in $L^1(X)$ if and only if

$$\forall g \in L^\infty(X) \lim_{n\to\infty} \int_X f_n(x) g(x) d\mu = \int_X f g d\mu.$$

We often write

$$f_n \overset{w}{\to} f \text{ in } L^1(X).$$

In general, weak convergence does not imply neither norm (i.e. strong) nor uniform, pointwise a.e. and in measure types of convergence.

The next well-known theorem characterizes the weak convergence in $L^1(X)$.

Theorem 1.3. Let $(X, \mathcal{F}, \mu)$ be a measure space, (f_n) a sequence in $L^1(X)$ and $f \in L^1(X)$. Then $f_n \overset{w}{\to} f$ in $L^1(X)$ if and only if

(i) $\forall A \in \mathcal{F},\ \int_A (f_n - f) d\mu \underset{n\to\infty}{\longrightarrow} 0$;

(ii) $\sup\{\|f_n\| : n \in \mathbb{N}\} = K < \infty$.

We will use the following result of Jean Dieudonné (see [3]).

Theorem 1.4. Let Y be a locally compact Hausdorff space, let $(Y, \mathcal{F}, \mu)$ be a measure space with a regular measure μ. Then a sequence (f_n) of integrable functions converges weakly in $L^1(Y)$ to an integrable function f if and only if $\forall A \in \mathcal{F}$ the limit

$$\lim_{n\to\infty} \int_A f_n(x)\, d\mu$$

exists and is finite.

1.2. Quasi-Young measures

Roughly speaking, the Young measure is a family of probability measures associated, in a sense, with measurable function(s) having values in a fixed compact subset of $\mathbb{R}^n$. The concept of Young measures has its roots in an oscillatory nature of minimizing sequences of bounded below integral functionals that do not attain their infima. The theory (and engineering practice) preparing background for Young measures is rather involved and we suggest that the reader consult [4] and [5] for details. A sketch of the theory, with main ideas and theorems without proofs, based on [5], can be also found in [2]. Here we propose a somewhat simpler object to consider, namely the *quasi-Young measure* (defined in [2]).

Let $a, b \in \mathbb{R}, a < b, \Omega := (a, b)$ and $K \subset \mathbb{R}$ be a compact set. Further, define $d\mu := \frac{1}{|b-a|}dx$ with Lebesgue measure dx. Denote by f a measurable function defined on Ω with values in K such that $\overline{f(\Omega)} = K$. The space of continuous functions from K to $\mathbb{R}$ will be denoted by $C(K, \mathbb{R})$.

Definition 1.5. We say that a family of probability measures $(\nu_x)_{x\in\Omega}$ is a quasi-Young measure associated with a measurable function f, if for every $\beta \in C(K, \mathbb{R})$ there holds an equality

$$\int_K \beta(k) d\nu_x(k) = \int_\Omega \beta(f(x)) d\mu. \tag{1}$$

Proposition 1.6. Let f be a strictly increasing function, differentiable on Ω and such that $\overline{f(\Omega)} = K := [c, d]$. Then a quasi-Young measure associated with f is a measure that is absolutely continuous with respect to the Lebesgue measure on K. Its density is equal to $\frac{1}{|b-a|}(f^{-1})'$.

Proof. Using the change of variable theorem we get

$$\int_c^d \beta(k)\, d\nu_x(k) = \int_a^b \beta(f(x))\, d\mu = \int_c^d \beta(y) \frac{1}{|b-a|}(f^{-1})'\,(y) dy. \qquad \blacksquare$$

Remark 1.7. Observe that the quasi-Young measure does not depend on the variable x in this case. Such a (quasi-)Young measure is called *homogeneous*.

Consider now the right-hand integral in the proof of Proposition 1.6. Recalling the properties of the Lebesgue-Stieltjes integral we see that we can write

$$\int_c^d \beta(y) \frac{1}{|b-a|}(f^{-1})'\,(y) dy = \int_c^d \beta(y) d\frac{1}{|b-a|}f^{-1}\,(y). \tag{2}$$

This shows that in the special case of real valued monotonic function defined on a one-dimensional interval the associated quasi-Young measure is in fact a Lebesgue-Stieltjes measure. This fact will be used in the next section.

2. Weak convergence in L^1

In this section we will assume for simplicity that $\Omega := (0,1)$, $K := [0,1]$. We can always return to the general case by scaling.

Theorem 2.1. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of strictly monotonic functions defined on Ω with values in K and such that:

(i) $\forall n \in \mathbb{N}\ \overline{u_n(\Omega)} = K$;

(ii) $\forall$n ∈ N the function u_n has continuously differentiable inverse $(u_n^{-1})'$;

(iii) the sequence $(u_n^{-1})'$ is nondecreasing.

Then the sequence $(u_n^{-1})'$ is weakly convergent in $L^1(K)$ to some $w \in L^1(K)$.

Proof. By assumption for any $m, n \in \mathbb{N}, m \leq n$ we have $(u_m^{-1})' \leq (u_n^{-1})'$, so for any $(u_m^{-1})' \leq (u_n^{-1})'$ and for any fixed Borel subset A of K we have

$$\int_A (u_m^{-1})'(y)dy \leq \int_A (u_n^{-1})'(y)dy,$$

so the sequence $\left(\int_A (u_n^{-1})'(y)dy\right)$ is monotonically increasing. It is also bounded due to the fact, that by (2) the quasi-Young measure associated with $(u_n^{-1})'$ is a regular probability measure on K. Thus for any fixed Borel subset A of K the limit

$$\lim_{n\to\infty} \int_A \left(u_n^{-1}\right)'(y)dy = \lim_{n\to\infty} \int_A du_n^{-1}(y)$$

exists and is finite which by Theorem 1.4 yields the result. ■

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