# RELATION BETWEEN DETERMINANTS OF TRIDIAGONAL AND CERTAIN PENTADIAGONAL MATRICES

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**Abstract.** In this paper we consider a pentadiagonal matrix which consists of only three non-zero bands. We prove that the determinant of such a matrix can be represented by a product of two determinants of corresponding tridiagonal matrices. It is shown that such an approach gives greatly shorter time of computer calculations.

***Keywords:*** *pentadiagonal matrix, tridiagonal matrix, determinant, recurrence equations*

## Introduction

This paper constitutes generalization of results presented in [1]. Let us consider a pentadiagonal matrix of the form

$$A_n = \begin{bmatrix} a_1 & 0 & c_1 & & & & & & & \\ 0 & a_2 & 0 & c_2 & & & & & & \\ b_3 & 0 & a_3 & 0 & c_3 & & & & & \\ & b_4 & 0 & a_4 & 0 & c_4 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & b_{n-3} & 0 & a_{n-3} & 0 & c_{n-3} & \\ & & & & & b_{n-2} & 0 & a_{n-2} & 0 & c_{n-2} \\ & & & & & & b_{n-1} & 0 & a_{n-1} & 0 \\ & & & & & & & b_n & 0 & a_n \end{bmatrix} \tag{1}$$

It means that $A_n = [a_{ij}]_{nxn}$ where $a_{ij} = 0$ for $|i-j| > 2$ and for $|i-j| = 1$. Determinant $W_n$ of matrix $A_n$ can be represented by a particular solution of fourth order linear recurrence equation (see [2])

$$W_{n+4} = a_{n+4}W_{n+3} - a_{n+3}b_{n+4}c_{n+2}W_{n+1} + b_{n+3}b_{n+4}c_{n+1}c_{n+2}W_n \tag{2}$$

with initial conditions of the form

$$\begin{cases} W_1 = a_1 \\ W_2 = a_1 a_2 \\ W_3 = a_3 W_2 - a_2 b_3 c_1 \\ W_4 = a_4 W_3 + b_4 c_2 (b_3 c_1 - a_1 a_3) \end{cases} \tag{3}$$

Now, let us introduce two auxiliary tridiagonal matrices of order $k$

$$A_k^* = \begin{bmatrix} a_1 & c_1 & & & & & \\ b_3 & a_3 & c_3 & & & & \\ & b_5 & a_5 & c_5 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & b_{2k-5} & a_{2k-5} & c_{2k-5} & \\ & & & & b_{2k-3} & a_{2k-3} & c_{2k-3} \\ & & & & & b_{2k-1} & a_{2k-1} \end{bmatrix} \tag{4}$$

and

$$A_k^{**} = \begin{bmatrix} a_2 & c_2 & & & & & \\ b_4 & a_4 & c_4 & & & & \\ & b_6 & a_6 & c_6 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & b_{2k-4} & a_{2k-4} & c_{2k-4} & \\ & & & & b_{2k-2} & a_{2k-2} & c_{2k-2} \\ & & & & & b_{2k} & a_{2k} \end{bmatrix} \tag{5}$$

By $W_k^*$, $W_k^{**}$ we denote the determinants of matrices $A_k^*$, $A_k^{**}$ respectively. Following [2], we can represent determinant $W_k^*$ by a particular solution of second order linear recurrence equation

$$W_{k+2}^* = a_{2k+3} W_{k+1}^* - b_{2k+3} c_{2k+1} W_k^* \tag{6}$$

with initial conditions of the form

$$\begin{cases} W_1^* = a_1 \\ W_2^* = a_1 a_3 - b_3 c_1 \end{cases} \tag{7}$$

In the same way we obtain recurrence relation for determinant $W_k^{**}$

$$W_{k+2}^{**} = a_{2k+4} W_{k+1}^{**} - b_{2k+4} c_{2k+2} W_k^{**} \tag{8}$$

where

$$\begin{cases} W_1^{**} = a_2 \\ W_2^{**} = a_2 a_4 - b_4 c_2 \end{cases} \tag{9}$$

## The main results

In this section we show the relationship between the determinant of matrix $A_n$ and determinants of matrices $A_k^*$, $A_k^{**}$.

**Theorem 1.** *Let* $A_n$, $A_k^*$, $A_k^{**}$ *be the matrices given by* (1), (4), (5) *and* $W_n$, $W_k^*$, $W_k^{**}$ *be their determinants respectively. Moreover, let us assume that* $W_0^{**} = 1$. *Then the following statements hold:*

1) *If* $n = 2k - 1$ *then*

$$W_n = W_{2k-1} = W_{k-1}^{**} \cdot W_k^*, \; k = 1,2,\ldots,\frac{n+1}{2} \tag{10}$$

2) *If* $n = 2k$ *then*

$$W_n = W_{2k} = W_k^{**} \cdot W_k^*, \; k = 1,2,\ldots,\frac{n}{2} \tag{11}$$

**Proof.** We are to prove that every determinant $W_n$, $n \in \mathbf{N}$, given by relationships (10) and (11) satisfies recurrence equation (2) with initial conditions (3).
Firstly, we prove that initial conditions (3) can be obtained from (10) and (11). Substituting consecutively $k = 1$, $k = 2$ into (10) and (11) we have

$$W_1 = W_0^{**} \cdot W_1^*$$
$$W_2 = W_1^{**} \cdot W_1^*$$
$$W_3 = W_1^{**} \cdot W_2^*$$
$$W_4 = W_2^{**} \cdot W_2^*$$

Simultaneously from (3), (7) and (9) under the assumption that $W_0^{**} = 1$ we obtain

$$W_1 = a_1 = 1 \cdot a_1 = W_0^{**} \cdot W_1^{*}$$
$$W_2 = a_1 \cdot a_2 = W_1^{*} \cdot W_1^{**}$$
$$W_3 = a_3 W_2 - a_2 b_3 c_1 = a_1 a_2 a_3 - a_2 b_3 c_1 = a_2 \left(a_1 a_3 - b_3 c_1\right) = W_1^{**} \cdot W_2^{*}$$
$$W_4 = a_4 W_3 + b_4 c_2 \left(b_3 c_1 - a_1 a_3\right) = a_2 a_4 \left(a_1 a_3 - b_3 c_1\right) - b_4 c_2 \left(a_1 a_3 - b_3 c_1\right) =$$
$$\left(a_2 a_4 - b_4 c_2\right)\left(a_1 a_3 - b_3 c_1\right) = W_2^{**} \cdot W_2^{*}$$

So, it was shown that initial conditions (3) can be obtained from (10) and (11). Secondly we are to show that every determinant $W_n$, $n > 4$, given by relationships (10) and (11) satisfies recurrence equation (2). Substituting $n = 2k - 1$ into (2) we have

$$W_{2k+3} = a_{2k+3} W_{2k+2} - a_{2k+2} b_{2k+3} c_{2k+1} W_{2k} + b_{2k+2} b_{2k+3} c_{2k+1} c_{2k} W_{2k-1} \tag{12}$$

At the same time from (10) and (11) we get

$$\begin{aligned} W_{2k+3} &= W_{k+1}^{**} W_{k+2}^{*} \\ W_{2k+2} &= W_{k+1}^{**} W_{k+1}^{*} \\ W_{2k} &= W_{k}^{**} W_{k}^{*} \\ W_{2k-1} &= W_{k-1}^{**} W_{k}^{*} \end{aligned} \tag{13}$$

Hence we have to show that equation

$$W_{k+1}^{**} W_{k+2}^{*} = a_{2k+3} W_{k+1}^{**} W_{k+1}^{*} - a_{2k+2} b_{2k+3} c_{2k+1} W_{k}^{**} W_{k}^{*} + b_{2k+2} b_{2k+3} c_{2k+1} c_{2k} W_{k-1}^{**} W_{k}^{*} \tag{14}$$

holds for every $k \in \mathbf{N}$.
We start with the left-hand side of equation (14). Bearing in mind (8) we obtain

$$\begin{aligned} W_{k+1}^{**} W_{k+2}^{*} &= \left(a_{2k+2} W_{k}^{**} - b_{2k+2} c_{2k} W_{k-1}^{**}\right)\left(a_{2k+3} W_{k+1}^{*} - b_{2k+3} c_{2k+1} W_{k}^{*}\right) = \\ &= a_{2k+2} a_{2k+3} W_{k}^{**} W_{k+1}^{*} - a_{2k+2} b_{2k+3} c_{2k+1} W_{k}^{**} W_{k}^{*} - a_{2k+3} b_{2k+2} c_{2k} W_{k-1}^{**} W_{k+1}^{*} + \\ &+ b_{2k+2} b_{2k+3} c_{2k} c_{2k+1} W_{k-1}^{**} W_{k}^{*} = a_{2k+3} W_{k+1}^{*} \left(a_{2k+2} W_{k}^{**} - b_{2k+2} c_{2k} W_{k-1}^{**}\right) + \\ &- a_{2k+2} b_{2k+3} c_{2k+1} W_{k}^{**} W_{k}^{*} + b_{2k+2} b_{2k+3} c_{2k} c_{2k+1} W_{k-1}^{**} W_{k}^{*} = \\ &= a_{2k+3} W_{k+1}^{*} W_{k+1}^{**} - a_{2k+2} b_{2k+3} c_{2k+1} W_{k}^{**} W_{k}^{*} + b_{2k+2} b_{2k+3} c_{2k} c_{2k+1} W_{k-1}^{**} W_{k}^{*} \end{aligned}$$

Let us observe that at the end of the above transformations we have obtained the right-hand side of equation (14).
Now, let us substitute $n = 2k$ into equation (2)

$$W_{2k+4} = a_{2k+4} W_{2k+3} - a_{2k+3} b_{2k+4} c_{2k+2} W_{2k+1} + b_{2k+3} b_{2k+4} c_{2k+2} c_{2k+1} W_{2k}$$

From (10) and (11) we get

$$\begin{aligned} W_{2k+4} &= W_{k+2}^{**}W_{k+2}^{*} \\ W_{2k+1} &= W_{k}^{**}W_{k+1}^{*} \end{aligned} \tag{15}$$

Moreover, from (13) we take the formulae for $W_{2k+3}$ and $W_{2k}$. Hence we must prove that the equation

$$W_{k+2}^{**}W_{k+2}^{*} = a_{2k+4}W_{k+1}^{**}W_{k+2}^{*} - a_{2k+3}b_{2k+4}c_{2k+2}W_{k}^{**}W_{k+1}^{*} + b_{2k+3}b_{2k+4}c_{2k+2}c_{2k+1}W_{k}^{**}W_{k}^{*} \tag{16}$$

holds for every $k \in \mathbf{N}$.
We start with the left-hand side of the equation (16). Bearing in mind (6) and (8) we have

$$\begin{aligned} W_{k+2}^{**}W_{k+2}^{*} &= \left(a_{2k+4}W_{k+1}^{**} - b_{2k+4}c_{2k+2}W_{k}^{**}\right)\left(a_{2k+3}W_{k+1}^{*} - b_{2k+3}c_{2k+1}W_{k}^{*}\right) = \\ &= a_{2k+3}a_{2k+4}W_{k+1}^{**}W_{k+1}^{*} - a_{2k+4}b_{2k+3}c_{2k+1}W_{k+1}^{**}W_{k}^{*} - a_{2k+3}b_{2k+4}c_{2k+2}W_{k}^{**}W_{k+1}^{*} + \\ &+ b_{2k+3}b_{2k+4}c_{2k+1}c_{2k+2}W_{k}^{**}W_{k}^{*} = a_{2k+4}\left(a_{2k+3}W_{k+1}^{*} - b_{2k+3}c_{2k+1}W_{k}^{*}\right)W_{k+1}^{**} + \\ &- a_{2k+3}b_{2k+4}c_{2k+2}W_{k}^{**}W_{k+1}^{*} + b_{2k+3}b_{2k+4}c_{2k+1}c_{2k+2}W_{k}^{**}W_{k}^{*} = \\ &= a_{2k+4}W_{k+2}^{*}W_{k+1}^{**} - a_{2k+3}b_{2k+4}c_{2k+2}W_{k}^{**}W_{k+1}^{*} + b_{2k+3}b_{2k+4}c_{2k+1}c_{2k+2}W_{k}^{**}W_{k}^{*} \end{aligned}$$

Hence we have obtained the right-hand side of equation (16), which ends the proof.

### Example

Let us consider a special form of pentadiagonal matrix (1) in which elements on diagonals are defined by sequences of the form $\left(a_k\right)_{k=1}^{n} = k$, $\left(b_k\right)_{k=3}^{n} = 2k$, $\left(c_k\right)_{k=1}^{n-2} = 2k-3$. Moreover, we assume that $n = 10^6$, i.e. the matrix has the order $10^6$. The value of the determinant of the considered matrix will be obtained in two ways.

Firstly we apply fourth order linear recurrence equation (2) with initial conditions (3), which in this case have the forms

$$W_{n+4} = (n+4)W_{n+3} - (n+3)(2n+8)(2n+1)W_{n+1} + (2n+6)(2n+8)(2n-1)(2n+1)W_n$$

$$W_1 = 1,\ \ W_2 = 2,\ \ W_3 = 18,\ \ W_4 = 0$$

where $n = 1,2,...,10^6 - 4$.

Let us observe that the above recurrence equation has functional coefficients, hence it is impossible to solve this equation using known analytical methods [3]. The proper algorithm for computing the determinant of this matrix will be implemented in the Maple system [4]. We apply the following steps:

$a := evalf([seq(n, n = 1..10^6)])$:
$b := evalf([0,0, seq(2n, n = 3..10^6)])$:
$c := evalf([seq(2n-3, n = 1..10^6)])$:
$W[1] := a[1]$:
$W[2] := a[1] \cdot a[2]$:
$W[3] := a[3] \cdot W[2] - a[2] \cdot b[3] \cdot c[1]$:
$W[4] := a[4] \cdot W[3] + b[4] \cdot c[2] \cdot (b[3] \cdot c[1] - a[1] \cdot a[3])$:
**for** $n$ **from** 1 **to** 999996 **do**
$W[n+4] := a[n+4] \cdot W[n+3] - a[n+3] \cdot b[n+4] \cdot c[n+2] \cdot W[n+1] +$
$b[n+3] \cdot b[n+4] \cdot c[n+1] \cdot c[n+2] \cdot W[n]$:
**end do**:
$print(evalf(W[1000000]))$

Finally we get $-6.43082825 \times 10^{5866732}$ as the value of determinant of the matrix under consideration.
Secondly, we apply Theorem 1, from which we have

$$W_{1000000} = W^{**}_{500000} \cdot W^{*}_{500000}$$

where determinants $W^*$, $W^{**}$ will be obtained from formulae (6), (7) and (8), (9). These formulae in this case have the following forms:

$$W^*_{k+2} = (2k+3)W^*_{k+1} - (4k+6)(4k-1)W^*_k$$

$$W^*_1 = 1,\ W^*_2 = 9$$

$$W^{**}_{k+2} = (2k+4)W^{**}_{k+1} - (4k+8)(4k+1)W^{**}_k$$

$$W^{**}_1 = 2,\ W^{**}_2 = 0$$

Let us denote $F = W^*$, $G = W^{**}$ and apply the following syntax:

$a := evalf([seq(n, n = 1..10^6)])$:
$b := evalf([0,0, seq(2n, n = 3..10^6)])$:
$c := evalf([seq(2n-3, n = 1..10^6)])$:
$F[1] := a[1]$:
$F[2] := a[1] \cdot a[3] - b[3] \cdot c[1]$:

**for** $k$ **from** 1 **to** 499998 **do**

$F[k+2] := a[2k+3]\cdot F[k+1] - b[2k+3]\cdot c[2k+1]\cdot F[k]:$

**end do**:

$G[1] := a[2]:$

$G[2] := a[2]\cdot a[4] - b[4]\cdot c[2]:$

**for** $k$ **from** 1 **to** 499998 **do**

$G[k+2] := a[2k+4]\cdot G[k+1] - b[2k+4]\cdot c[2k+2]\cdot G[k]:$

**end do**:

$print(evalf(F[500000]\cdot G[500000]))$

Finally we get $-6.430935419\times 10^{5866732}$ as the value of determinant of the matrix under consideration. The presented results were obtained with the Maple default precision (Digits = 10). It has to be emphasized that the running time of the first algorithm was 40 s, whilst of the second was 17 s.

## Conclusions

It was shown that the determinant of the pentadiagonal matrix, which consists of only three non-zero bands, can be represented by a product of two determinants of corresponding tridiagonal matrices. This means that this determinant is obtained as a product of particular solutions of two second order homogeneous linear recurrence equations.

In the presented example we dealt with a $10^6\times 10^6$ pentadiagonal matrix. In order to obtain the determinant of the considered matrix we used two approaches. Firstly we had a formulated algorithm based on proper fourth order homogeneous linear recurrence equation. In second algorithm we used the theorem proved in this paper. Both algorithms were implemented in the Maple system. It turned out that time of calculations was shorter in the second approach.

## References

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