DIFFERENTIATION AND INTEGRATION
BY USING MATRIX INVERSION

Dagmara Matlak, Jarosław Matlak, Damian Słota, Roman Witula

Institute of Mathematics, Silesian University of Technology, Gliwice, Poland
roman.witula@polsl.pl

Abstract. In the paper certain examples of applications of the matrix inverses for generating and calculating the integrals are presented.

Keywords: matrix inverse, integrals, generalized McLaurin’s formula

Introduction

The first part of our discussion concerns the linear mappings defined on the finite-dimensional space of solutions of the following system of differential equations

\[

d_1 f'_1 = a_{1,1} f_1 + \cdots + a_{1,n} f_n,

d_2 f'_2 = a_{2,1} f_1 + \cdots + a_{2,n} f_n,
\]

\[
\vdots
\]

\[

d_n f'_n = a_{n,1} f_1 + \cdots + a_{n,n} f_n.
\]

(1)

Suppose that functions \(g_1, g_2, \ldots, g_n\) form the solution of the above system of equations and matrix \(A = [a_{i,j}]_{n \times n}\) of system (1) is nonsingular. Let us consider the linear mapping \(T\) of the linear space of solutions \((f_1, f_2, \ldots, f_n)^T\) of system (1) onto itself defined in the following way:

\[
T\left(\begin{array}{c}
d_1 f'_1 \\
d_2 f'_2 \\
\vdots \\
d_n f'_n
\end{array}\right) = \left(\begin{array}{c}
d_1 f'_1 + \cdots + a_{1,n} f_n \\
d_2 f'_2 + \cdots + a_{2,n} f_n \\
\vdots \\
\vdots \\
d_n f'_n + \cdots + a_{n,n} f_n
\end{array}\right) = \left[\begin{array}{ccc}
a_{1,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots \\
a_{n,1} & \cdots & a_{n,n}
\end{array}\right]
\left(\begin{array}{c}
f'_1 \\
f'_2 \\
\vdots \\
f'_n
\end{array}\right) = A
\left(\begin{array}{c}
f'_1 \\
f'_2 \\
\vdots \\
f'_n
\end{array}\right).
\]

(2)

Matrix \(A\) is nonsingular, so there exists its inverse \(A^{-1}\). In particular, the following equality occurs:

\[
T\left(\begin{array}{c}
A^{-1} g_1 \\
\vdots \\
A^{-1} g_n
\end{array}\right) = AA^{-1}\left(\begin{array}{c}
g_1 \\
\vdots \\
g_n
\end{array}\right).
\]

(3)
Therefore, if \( A^{-1} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} G_1 \\ \vdots \\ G_n \end{pmatrix} \), then functions \( G_i \) are the primitive functions of \( g_i \) (\( i = 1, 2, \ldots, n \)).

The current article is inspired by Swartz’s paper [1] where the author gives some simple examples of using this procedure, among others, for generating the integrals of functions \( h_1 = e^{ax} \sin bx \), \( h_2 = e^{ax} \cos bx \), for which he obtained the formula (the integration constants are omitted and this rule will oblige henceforward):

\[
\begin{pmatrix}
\int h_1 \, dx \\
\int h_2 \, dx
\end{pmatrix} = \begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} \begin{pmatrix}
h_1 \\
h_2
\end{pmatrix} = \frac{e^{ax}}{a^2 + b^2} \left( a \sin bx - b \cos bx \right).
\]

(4)

1. Generalization of Swartz’s example

Let us start from the generalization of the example mentioned above. Let us take the functions

\[
\begin{align*}
g_1(x) &= \cosh ax \sin bx, \\
g_2(x) &= \cosh ax \cos bx, \\
g_3(x) &= \sinh ax \cos bx, \\
g_4(x) &= \sinh ax \sin bx.
\end{align*}
\]

(5)

Note that the differentiation operator for these functions is of the form

\[
T \begin{pmatrix} 
g_1 \\
g_2 \\
g_3 \\
g_4
\end{pmatrix} = \begin{pmatrix} 
g_1' \\
g_2' \\
g_3' \\
g_4'
\end{pmatrix} = \begin{bmatrix}
0 & b & 0 & a \\
-b & 0 & a & 0 \\
a & 0 & b & 0
\end{bmatrix} \begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4
\end{pmatrix}.
\]

(6)

If \( ab \neq 0 \), then the inverse of matrix of operator \( T \) has the form

\[
A^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix}
0 & -b & 0 & a \\
b & 0 & a & 0 \\
0 & a & 0 & b \\
a & 0 & -b & 0
\end{bmatrix}.
\]

(7)

Now we can easily integrate functions \( g_i \), e.g.

\[
\int g_1(x) \, dx = \frac{-bg_2(x) + ag_4(x)}{a^2 + b^2}.
\]
2. Integrals of \( \sin^n x \) for odd \( n \)

Consider the second derivatives of functions \( g(x) = \sin^n x, n \geq 2 \). We have

\[
(sin^n x)'' = (n \sin^{n-1} x \cos x)' = n(n - 1) \sin^{n-2} x - n^2 \sin^n x.
\] (8)

Of course for \( n = 1 \) there is \( (\sin x)'' = -\sin x \). Thus we can write the second derivative operator for the odd powers of function \( \sin x \), from 1 to odd \( 2n + 1 \), in the following matrix form:

\[
A_k \begin{pmatrix} \sin x \\ \sin^3 x \\ \vdots \\ \sin^k x \end{pmatrix} = \begin{pmatrix} (\sin x)'' \\ (\sin^3 x)'' \\ \vdots \\ (\sin^k x)'' \end{pmatrix} = \begin{pmatrix} -\sin x \\ 6 \sin x - 9 \sin^3 x \\ \vdots \\ k(k - 1) \sin^{k-2} x - k^2 \sin^k x \end{pmatrix} =
\]

\[
\begin{bmatrix}
-1^2 & 0 & \cdots & 0 & 0 \\
3 \cdot 2 & -3^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & k(k - 1) & -k^2
\end{bmatrix}
\begin{pmatrix}
\sin x \\
\sin^3 x \\
\vdots \\
\sin^k x
\end{pmatrix}.
\]

(9)

The determinant of the obtained matrix is equal to: \( \det A_k = (-1)^\frac{k+1}{2} (k!!)^2 \neq 0 \), \( k \in \mathbb{N} \). The inverse matrix is of the form

\[
A_k^{-1} = \left[ a_{i,j} \right]_{\frac{k+1}{2} \times \frac{k+1}{2}}
\]

where

\[
a_{i,j} = \begin{cases} 
0, & i < j, \\
-\frac{1}{(2l-1)^2}, & i = j, \\
-\frac{(n-1)!!}{n!!} \frac{1}{(2l-1)(2l-2)!!}, & i > j.
\end{cases}
\]

(11)

We can deduce that for odd \( n \) there occurs (with respect to the linear element):

\[
\int_0^2 \sin^n x \, dx = -\frac{(n-1)!!}{n!!} \sum_{l=0}^{(n-1)/2} \frac{1}{2l+1} \frac{(2l-1)!!}{(2l)!!} \sin^{2l+1} x,
\]

(12)

where \( \int f(x) \, dx = \int (\int f(x) \, dx) \, dx \). For example, we get

\[
\int_0^2 \sin^5 x \, dx = -\frac{8}{15} \sin x - \frac{4}{45} \sin^3 x - \frac{1}{25} \sin^5 x.
\]

(13)

We note that from (12) by differentiating we obtain (see [2, 3]):

\[
\int \sin^n x \, dx = -\frac{(n-1)!!}{n!!} \cos x \sum_{l=0}^{(n-1)/2} \frac{(2l-1)!!}{(2l)!!} \sin^{2l} x.
\]

(14)
For example, we have

\[ \int \sin^5 x \, dx = -\frac{8}{15} \cos x \, (1 + \frac{1}{2} \sin x + \frac{3}{8} \sin^4 x). \]

### 3. The case of the even powers of \( \sin x \)

Consider functions of the form

\[ g_n(x) = \sin^n x - \frac{n-1}{n} \sin^{n-2} x, \tag{15} \]

for \( n = 2, 4, 6, \ldots \). Acting on the vector \( \begin{pmatrix} g_2(x) \\ g_4(x) \\ \vdots \\ g_k(x) \end{pmatrix} \), where \( k \) is even, with the second derivative operator, like it was done in equation (9), we get the following transformation matrix:

\[
B_k \begin{pmatrix} g_2(x) \\ g_4(x) \\ \vdots \\ g_k(x) \end{pmatrix} = \begin{pmatrix} g_2(x)'' \\ g_4(x)'' \\ \vdots \\ g_k(x)'' \end{pmatrix} = \begin{bmatrix}
-2^2 & 0 & \cdots & 0 & 0 \\
2^2 \cdot 3/4 & -4^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & (k-2)^2(k-1)/k & -k^2
\end{bmatrix} \begin{pmatrix} g_2(x) \\ g_4(x) \\ \vdots \\ g_k(x) \end{pmatrix}. \tag{16}
\]

The above matrix is invertible and its inversion is of the form

\[
B_k^{-1} = \begin{pmatrix} b_{i,j} \end{pmatrix}_{k \times k}, \tag{17}
\]

where

\[
b_{i,j} = \begin{cases} 
0, & i < j, \\
-\frac{1}{(2i)!^2}, & i = j, \\
-\frac{1}{(2i)!^2} \frac{(2j)!}{(2j-1)!} & i > j.
\end{cases} \tag{18}
\]

Therefore for even \( n \) we get the formula (exact to the linear element):

\[
\int^2 g_n(x) \, dx = -\frac{(n-1)!}{n!n^2} \sum_{i=1}^{n/2} \frac{(2i)!}{(2i-1)!} \left( \sin^{2i} x - \frac{2i-1}{2i} \sin^{2i-2} x \right) = -\frac{(n-1)!}{n!n^2} \left( \sin^n x - 1 + \sum_{i=1}^{n/2-1} \sin^{2i} x \left( \frac{(2i)!}{(2i-1)!} - \frac{(2i+2)!}{(2i+1)!} \right) \right) = \frac{(n-1)!}{n!n^2} - \frac{1}{n^2} \sin^n x = -\frac{1}{n^2} \sin^n x, \tag{19}
\]

or
Differentiation and integration by using matrix inversion

which implies the following integral identity

\[
\int^2 \sin^n x \, dx = \frac{(n-1)!}{n!!} \left( \int^2 dx + \sum_{k=1}^{n/2} \frac{(2k-2)!}{(2k-1)!} \int^2 g_{2k}(x) \, dx \right)

= \frac{(n-1)!}{n!!} \left( \frac{x^2}{2} - \sum_{k=1}^{n/2} \frac{(2k-2)!}{(2k-1)!} \frac{x^{2k-1}}{(2k-1)!} \sin^{2k-1} x \right).
\]  

(20)

Hence, by differentiating we get (see [2, 3]):

\[
\int \sin^n x \, dx = \frac{(n-1)!}{n!!} \left( x - (\cos x) \sum_{k=1}^{n/2} \frac{(2k-2)!}{(2k-1)!} \sin^{2k-1} x \right).
\]  

(21)

For example, we obtain

\[
\int \sin^n x \, dx = \frac{15}{48} \left( x - \cos x \left( \sin x + \frac{2}{3} \sin^3 x + \frac{6}{15} \sin^5 x \right) \right).
\]  

(22)

4. Integral of \( \tan^n x \)

Let \( V \) be the linear space of sequences \( \{f_n(x)\}_{n=0}^\infty \) of differentiable functions \( f_n: (a, b) \to R \). Let \( \mathbb{A}: V \to V \) be a linear operator satisfying equation

\[
\begin{pmatrix}
(tan x)' \\
(tan^2 x)' \\
(tan^3 x)' \\
\vdots
\end{pmatrix} = \mathbb{A} \begin{pmatrix}
1 \\
\tan x \\
\tan^2 x \\
\vdots
\end{pmatrix}.
\]  

(23)

If \( \mathbb{A} \) is represented by infinite matrix \( A \), then from (23) matrix \( A \) has the form

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 2 & 0 & 2 & 0 & \cdots \\
0 & 0 & 3 & 0 & 3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]  

(24)

It is easy to show that

\[
A^{-1} = \begin{bmatrix}
1 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\
0 & 1 & 0 & \frac{1}{3} & 0 & \cdots \\
0 & 0 & 1 & 0 & \frac{1}{3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]  

(25)

However, matrix \( A^{-1} \) does not represent the inverse operator \( \mathbb{A}^{-1} \), since the following relations hold
\[
\begin{pmatrix}
(tan x)'
(tan^2 x)'
(tan^3 x)'
\vdots
\end{pmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \vdots \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & \vdots \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\begin{pmatrix}
tan x \\
tan^2 x \\
tan^3 x \\
\vdots
\end{pmatrix},
\]
(26)

resulting from the formula
\[
d \frac{d}{dx} \left( \frac{1}{n+1} tan^{n+1} x - \frac{1}{n+3} tan^{n+3} x \right) = tan^n x - tan^{n+4} x.
\]
(27)

Moreover, we get from this, after summing over powers in the interval \([-\pi/4, \pi/4]\) and by uniform convergence (see [4]), that
\[
tan^n x = \frac{d}{dx} \sum_{k=0}^{\infty} \left( \frac{1}{n+4k+1} tan^{n+4k+1} x - \frac{1}{n+4k+3} tan^{n+4k+3} x \right),
\]
or equivalently
\[
\int_0^x tan^n y dy = \sum_{k=0}^{\infty} \left( \frac{1}{n+2k+1} tan^{n+2k+1} x \right)
= \sum_{k=0}^{\infty} \frac{(-1)^k}{n+2k+1} tan^{n+2k+1} x,
\]
(29)

for every \(x \in [-\pi/4, \pi/4]\) and \(n = 0, 1, 2, \ldots\).

From formula (28) we also get the matrix form \(I(\mathbb{A})\) of operator \(\mathbb{A}^{-1}\), i.e. the inverse operator of operator \(\mathbb{A}\) (under assumption of its existence):
\[
I(\mathbb{A}) =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \vdots \\
0 & \frac{1}{2} & 0 & 0 & 0 & \vdots \\
-\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \vdots \\
0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \vdots \\
\frac{1}{5} & 0 & -\frac{1}{5} & 0 & \frac{1}{5} & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.
\]
(30)

Formula (29) is our main analytic result in this section. Why do we think so? Because, as we show now, this formula is a generalization of the classical MacLaurin’s formulae for \(\ln(x + 1)\), i.e.
\[
\ln(x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,
\]
(31)
for \(-1 < x \leq 1\), found independently by Nicolaus Mercator and Saint-Vincent (see sections 10-9 and 10-10 in [5] and page 387 in [6]), and for \(\text{arctan}\ x\), i.e.
\[
\text{arctan}\ x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots,
\]
(32)
for \(-1 < x \leq 1\), which is known as the Gregory series.
This connection should not be surprising because of the known complex relation (see [7]):

$$\arctan z = \frac{1}{2} \ln \left( \frac{1 + iz}{1 - iz} \right),$$  \hspace{1cm} (33)

where $z/i \in \mathbb{C} \setminus (-\infty, -1) \cup [1, \infty)$ and where the principal branch of the logarithm is under consideration. On the cuts we have

$$\arctan(iy) = \pm \frac{\pi}{2} + \frac{i}{2} \ln \left( \frac{y + i}{y - i} \right),$$  \hspace{1cm} (34)

for $y \in (-\infty, -1) \cup (1, \infty)$ and where the upper/lower sign corresponds to the right/left side of the set determining $y$. More precisely, the connection between the arctan function and log function is obvious and the section concerns the real and imaginary parts of $\arctan z$, since we have

$$\arctan z = \frac{1}{2} \arctan \left( \frac{2x}{1-x^2-y^2} \right) + \frac{1}{4} i \ln \left( \frac{x^2+(y+1)^2}{x^2+(y-1)^2} \right),$$  \hspace{1cm} (35)

where $z = x + iy$, $|z| < 1$. First, from equation (29) for $n = 1$ we get

$$-\ln \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2(k+1)} \tan^{2(k+1)} x$$  \hspace{1cm} (36)

or

$$\ln \cos(\arctan \sqrt{x}) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-x)^k}{k},$$  \hspace{1cm} (37)

which by (31) implies the well known identity (since $\cos \alpha = \frac{1}{\sqrt{1+\tan^2 \alpha}}$ for $\alpha \in (-\frac{\pi}{4}, \frac{\pi}{4})$):

$$\ln \cos(\arctan \sqrt{x}) = -\frac{1}{2} \ln(x + 1),$$  \hspace{1cm} (38)

i.e.

$$\sqrt{x + 1} \cos(\arctan x) \equiv 1,$$  \hspace{1cm} (39)

for every $x \in [0,1]$.

But this formula holds for every $x \geq 0$ since $\cos \alpha = \frac{1}{\sqrt{1+\tan^2 \alpha}}$ for every $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. In other words, formula (37) is equivalent to (31). For $n = 0$ from (29) we get

$$\frac{\arctan x}{x} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{2k+1},$$  \hspace{1cm} (40)
which implies (32). Hence, for $x = 1$ we obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(41)

That is the classical Gregory-Leibniz-Nilakantha’s formula (see [8]). Generalizations of the Gregory power series (32) are discussed in papers [9] and [10].

As seen from equation (29), the values of integrals $\int_0^\pi \tan^n x \, dx$ for $n \geq 2$ are the translations of numbers $\int_0^\pi \tan x \, dx$ or $\int_0^\pi x \, dx$, depending on parity of $n$. For even $n$ we have

$$\int_0^\pi \tan^n x \, dx = \int_0^\pi \tan x \, dx - \sum_{k=0}^{(n-2)/2} (-1)^k \frac{\pi}{2k+1} = \frac{\pi}{4} - \sum_{k=0}^{(n-2)/2} \frac{(-1)^k}{2k+1},$$

(42)

whereas for odd $n$ we get

$$\int_0^\pi \tan^n x \, dx = \int_0^\pi \tan x \, dx - \sum_{k=1}^{(n-1)/2} (-1)^k \frac{\pi}{2k} = \frac{1}{2} \left( \ln 2 - \sum_{k=1}^{(n-1)/2} \frac{(-1)^k}{2k} \right).$$

(43)

5. Final remark

Some other applications of the matrix obtained by $n$-times differentiation of product functions and composition functions are discussed in paper [11]. In turn, in paper [12] the technique of the inverse matrix was used for calculating the integral $\int \sec^{2n+1} x \, dx$, similarly as in the present study. The obtained formulae were used there for generating the trigonometric identities.

References


[12] Witula R., Matlak D., Matlak J., Slota D., Use of matrices in evaluation of $\int \sec^{2n+1} x \, dx$ in review.