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A GEOMETRIC PROPERTY OF THE ROOTS OF CHEBYSHEV POLYNOMIALS

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Abstract. In this text a new property of geometric nature of the Chebyshev polynomials is given. It is proven that the lengths of diagonals of a regular n-gon with the side of length equal to one are the sums of positive roots of the respective renormalized Chebyshev polynomials of one from among four types. Some new special decompositions of differences of values of the Chebyshev polynomials are also presented.

Keywords: *Chebyshev polynomials, Vieta-Lucas polynomials*

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Introduction

The aim of this short text is to present one more fundamental property of Chebyshev polynomials related to the geometric-analytical nature of their roots. We want to emphasize that this property seems to be completely original.

Inspired by paper [1], we have noticed a slightly deeper relationship between the length of diagonals of the regular n-gons and the sums of positive roots of any of all four types of Chebyshev polynomials (see [2-4]):

$$1st \text{ kind} \quad \boldsymbol{T}_n(\cos\, x) = \cos\,(n\, x), \qquad x \in \mathbb{R},$$

$$2nd \text{ kind} \quad \boldsymbol{U}_n(\cos\, x) = \frac{\sin\left((n+1)x\right)}{\sin\, x}, \qquad x \in \mathbb{R} \setminus \pi\mathbb{Z},$$

$$3rd \text{ kind} \quad \boldsymbol{V}_n(\cos\, x) = \frac{\cos\left(\left(n+\frac{1}{2}\right)x\right)}{\cos\frac{x}{2}}, \qquad x \in \mathbb{R} \setminus \pi(2\mathbb{Z}+1),$$

$$4th \text{ kind} \quad \boldsymbol{W}_n(\cos\, x) = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}}, \qquad x \in \mathbb{R} \setminus 2\pi\mathbb{Z},$$

for every $n \in \mathbb{N} \cup \{0\}$. For the reasons of cosmetic nature it is better to consider the re-scaling Chebyshev polynomials

$$\boldsymbol{T}_n^*(x) := 2\boldsymbol{T}_n\left(\frac{x}{2}\right), \quad \boldsymbol{U}_n^*(x) := \boldsymbol{U}_n\left(\frac{x}{2}\right), \quad \boldsymbol{V}_n^*(x) := \boldsymbol{V}_n\left(\frac{x}{2}\right),$$
$$\text{and} \quad \boldsymbol{W}_n^*(x) := \boldsymbol{W}_n\left(\frac{x}{2}\right), \quad n \in \mathbb{N}.$$

We note that polynomials $\boldsymbol{T}_n^*(x)$ are called either Vieta-Lucas polynomials [5, 6] or Dickson's polynomials [7]. Whereas $\boldsymbol{U}_n^*(x)$ are called Vieta-Fibonacci polynomials [6]. Properties of algebraic and combinatoric nature of these polynomials are discussed for example in papers [5-11]. One of the spectacular properties of $\boldsymbol{T}_n^*(x)$ are the following decompositions proved in [6]:

$$\begin{aligned}\boldsymbol{T}_{2n-1}^*(x) - \boldsymbol{T}_{2n-1}^*(\theta + \theta^{-1}) &= \boldsymbol{T}_{2n-1}^*(x) - \theta^{2n-1} - \theta^{-2n+1} = \\ &= \prod_{k=0}^{2n-2} (x - \theta\xi^{2k} - \theta^{-1}\xi^{-2k}),\end{aligned} \tag{1}$$

where $\xi = \exp\left(\frac{i\pi}{2n-1}\right)$;

$$\begin{aligned}(-1)^n\boldsymbol{T}_{2n}^*(i\,x) + \boldsymbol{T}_{2n}^*(\theta + \theta^{-1}) &= (-1)^n\boldsymbol{T}_{2n}^*(i\,x) + \theta^{2n} + \theta^{-2n} = \\ &= \prod_{k=0}^{2n-1} (x - \theta\zeta^{2k+1} + \theta^{-1}\zeta^{-2k-1}),\end{aligned} \tag{2}$$

where $\zeta = \exp\left(\frac{i\pi}{2n}\right)$, for every $n \in \mathbb{N}$, $\theta \in \mathbb{C}$ and $\theta \neq 0$. For example, we obtain the following special ones

$$\boldsymbol{T}_{2n-1}^*(x) - 2\cos\left((2n-1)\varphi\right) = \prod_{k=0}^{2n-2}\left(x - 2\cos\left(\varphi + \frac{2k\pi}{2n-1}\right)\right),$$

$$\begin{aligned}&\boldsymbol{T}_{2n-1}^*(x) - \boldsymbol{T}_{2n-1}^*(2\csc\varphi) = \boldsymbol{T}_{2n-1}^*(x) - \tan^{2n-1}\left(\frac{\varphi}{2}\right) - \cot^{2n-1}\left(\frac{\varphi}{2}\right) = \\ &= \prod_{k=0}^{2n-2}\left(x - 2\csc\varphi\cos\frac{2k\pi}{2n-1} + 2\,i\cot\varphi\sin\frac{2k\pi}{2n-1}\right),\end{aligned}$$

for $\varphi \neq k\,\pi$, $k \in \mathbb{Z}$,

$$\begin{aligned}&(-1)^n\,\boldsymbol{T}_{2n}^*(i\,x) + \boldsymbol{T}_{2n}^*(2\sec\alpha) = \\ &= (-1)^n\,\boldsymbol{T}_{2n}^*(i\,x) + \tan^{2n}\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \tan^{2n}\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = \\ &= \prod_{k=0}^{2n-1}\left(x + 2\tan\alpha\cos\frac{(2k+1)\pi}{2n} - 2\,i\sec\alpha\sin\frac{(2k+1)\pi}{2n}\right),\end{aligned}$$

for $\alpha \neq \frac{\pi}{2} + k\,\pi$, $k \in \mathbb{Z}$,

$$
\begin{aligned}
&(-1)^n\,\boldsymbol{T}^*_{2n}(i\,x) + \boldsymbol{T}^*_{2n}(\coth\ \alpha) = \\
&= (-1)^n\,\boldsymbol{T}^*_{2n}(i\,x) + \coth^{2n}\left(\frac{\alpha}{2}\right) + \tanh^{2n}\left(\frac{\alpha}{2}\right) = \\
&= \prod_{k=0}^{2n-1}\left(x + 2\ \operatorname{csch}\ \alpha\ \cos\ \frac{(2k+1)\pi}{2n} - 2\,i\ \coth\ \alpha\ \sin\ \frac{(2k+1)\pi}{2n}\right),
\end{aligned}
$$

for $\alpha \neq 0$.

Other fundamental properties, especially of an analytic nature, of polynomials $\boldsymbol{T}^*_n$, $\boldsymbol{U}^*_n$, $\boldsymbol{V}^*_n$ and $\boldsymbol{W}^*_n$ are presented in monographs [2, 4], see also interesting new results on a moment problem [12].

1. Main result

Positive roots of polynomials $\boldsymbol{T}^*_n$, $\boldsymbol{U}^*_n$, $\boldsymbol{V}^*_n$ and $\boldsymbol{W}^*_n$, respectively, are listed below [2]:

$$
\begin{aligned}
t_{k,n} &= 2\cos\left(\frac{(2k-1)\pi}{2n}\right), && k = 1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor, \\
u_{k,n} &= 2\cos\left(\frac{k\pi}{n+1}\right), && k = 1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor, \\
v_{k,n} &= 2\cos\left(\frac{(2k-1)\pi}{2n+1}\right), && k = 1,2,\dots,\left\lfloor\frac{n+1}{2}\right\rfloor, \\
w_{k,n} &= 2\cos\left(\frac{2k\pi}{2n+1}\right), && k = 1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor,
\end{aligned}
$$

(we note that zeros $t_{k,n}$, $k = 1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor$, could also be deduced from (1) for $\theta = \exp\left(\frac{i\pi}{4n-2}\right)$ and from (2) for $\theta = \exp\left(\frac{i\pi}{4n}\right)$).

We will present now our main result.

Theorem. *Let $n \in \mathbb{N}$, $n \geq 4$, and $V_0, V_1, \dots, V_{n-1}$ denote the vertices of a regular n-gon P with the side of length 1. If $x = \frac{\pi}{n}$ and $d(V_k, V_l)$ denotes the distance between V_k and V_l, then we have*

(i)

$$
d(V_0, V_m) = \frac{\sin(mx)}{\sin x} = \boldsymbol{U}_{m-1}(\cos x)
$$

for $1 \leq m \leq \left\lfloor\frac{n}{2}\right\rfloor$,

(ii)

$$d(V_0, V_{m+1}) - d(V_0, V_m) = \frac{\cos\left(\left(m+\frac{1}{2}\right)x\right)}{\cos\frac{x}{2}} = \boldsymbol{V}_m(\cos x),$$

for $1 \leq m+1 \leq \left\lfloor \frac{n}{2} \right\rfloor$,

(iii)

$$d(V_0, V_{m+1}) - d(V_0, V_{m-1}) = 2\cos(mx) = 2\boldsymbol{T}_m(\cos x),$$

for $2 \leq m+1 \leq \left\lfloor \frac{n}{2} \right\rfloor$,

(iv)

$$d(V_0, V_{2m}) = 2\sum_{k=1}^{m} \cos\big((2k-1)x\big),$$

for $2 \leq 2m \leq \left\lfloor \frac{n}{2} \right\rfloor$,

(v)

$$d(V_0, V_{2m+1}) = 1 + 2\sum_{k=1}^{m} \cos(2kx),$$

for $1 \leq 2m+1 \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Corollary. *By comparing the terms of sums from identities* (iv) *and* (v) *with the roots of all four types of rescaling Chebyshev polynomials* $\boldsymbol{T}_n^*$, $\boldsymbol{U}_n^*$, $\boldsymbol{V}_n^*$ *and* $\boldsymbol{W}_n^*$, *we obtain*

$$d(V_0, V_{2m}) = \begin{cases} \sum_{k=1}^{m} t_{k,r} & \text{if } n = 2r, \\ \sum_{k=1}^{m} v_{k,r} & \text{if } n = 2r+1, \end{cases}$$

for $2 \leq 2m \leq \left\lfloor \frac{n}{2} \right\rfloor$ *and*

$$d(V_0, V_{2m+1}) = \begin{cases} 1 + \sum_{k=1}^{m} w_{k,r} & \text{if } n = 2r+1, \\ 1 + \sum_{k=1}^{m} u_{k,r} & \text{if } n = 2(r+1), \end{cases}$$

for $1 \leq 2m+1 \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Proof of Theorem. If r denotes the radius of the circle circumscribed on P, then by the law of sines we get $r = \frac{1}{2\sin x}$ and $d(V_0, V_m) = \frac{\sin(mx)}{\sin x}$, $m \le \left\lfloor \frac{n}{2} \right\rfloor$. Hence we obtain

$$
\begin{aligned}
d(V_0, V_{m+1}) - d(V_0, V_m) &= \frac{\sin\big((m+1)x\big) - \sin(mx)}{\sin x} = \\
&= \frac{2\sin\frac{x}{2}\cos\left(\left(m+\frac{1}{2}\right)x\right)}{\sin x} = \frac{\cos\left(\left(m+\frac{1}{2}\right)x\right)}{\cos\frac{x}{2}} = \boldsymbol{V}_m(\cos x),
\end{aligned}
$$

for $1 \le m+1 \le \left\lfloor \frac{n}{2} \right\rfloor$, and next we have

$$
\begin{aligned}
& d(V_0, V_{m+1}) - d(V_0, V_{m-1}) = \\
&= (d(V_0, V_{m+1}) - d(V_0, V_m)) + (d(V_0, V_m) - d(V_0, V_{m-1})) = \\
&= \frac{\cos\left(\left(m+\frac{1}{2}\right)x\right) + \cos\left(\left(m-\frac{1}{2}\right)x\right)}{\cos\frac{x}{2}} = \\
&= \frac{2\cos(mx)\cos\frac{x}{2}}{\cos\frac{x}{2}} = 2\cos(mx) = 2\boldsymbol{T}_m(\cos x),
\end{aligned}
$$

for $2 \le m+1 \le \left\lfloor \frac{n}{2} \right\rfloor$.

The last identity implies

$$
d(V_0, V_{2m}) = \sum_{k=1}^{m} \left(d(V_0, V_{2k}) - d(V_0, V_{2(k-1)}) \right) = 2\sum_{k=1}^{m} \cos\big((2k-1)x\big),
$$

for $2 \le 2m \le \left\lfloor \frac{n}{2} \right\rfloor$, and

$$
d(V_0, V_{2m+1}) = 1 + \sum_{k=1}^{m} \left(d(V_0, V_{2k+1}) - d(V_0, V_{2k-1}) \right) = 1 + 2\sum_{k=1}^{m} \cos(2kx),
$$

for $2 \le 2m+1 \le \left\lfloor \frac{n}{2} \right\rfloor$, which ends the proof. ■

Conclusions

In the paper we have described the new property of zeros of renormalized Chebyshev polynomials. It has been proven that the lengths of diagonals of a regular *n*-gon with the side of length equal to one are the sums of positive roots of the respective renormalized Chebyshev polynomials of one from among four types. Formulae for decompositions of differences of values of the Chebyshev polynomials have been presented as well.

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