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ABOUT SOME PROPERTIES OF JACOBIAN FOR POLYNOMIAL MAPPINGS OF TWO COMPLEX VARIABLES

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Abstract. In our article we consider jacobian $Jac(f,h)$ of polynomial mapping $f = X^k Y^k + \ldots + f_1$, $h = X^{k-1} Y^{k-1} + \ldots + h_1$. We give conditions for coordinate h in which constant jacobian $Jac(f,h) = Jac(f_1,h_1)$ vanishes.

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Introduction

One of the most interesting issues of classical algebraic geometry is the question of polynomial mapping of two complex variables with constant jacobian zeros at infinity quantity [1]. In article [2] the authors showed that this kind of mapping has at most two zeros at infinity. In our article we give the conditions in which mapping in constant jacobian has one zero at infinity.

1. Selected properties of jacobians

To simplify the properties we assume that f and g are forms.

Property 1

We have following formulas:

1.1. Formula

$$Jac(X^pY^q, X^rY^s) = \begin{vmatrix} p & q \\ r & s \end{vmatrix} X^{p+r-1}Y^{q+s-1} \tag{1}$$

1.2. Formula

$$Jac(X^pY^pf, X^qY^qg) = \\ pX^{p+q-1}Y^{p+q-1}fJac(XY,g) - \\ qX^{p+q-1}Y^{p+q-1}gJac(XY,f) + \\ X^{p+q}Y^{p+q}Jac(f,g) \tag{2}$$

1.3. If *f* and *g* are *2k-1*-degree, then

$$Jac(XY,f) = Jac(XY,g) \Leftrightarrow f = g \tag{3}$$

1.4. If *f* and *g* are *2k*-degree forms, then

$$Jac(XY,f) = Jac(XY,g) \Leftrightarrow f = g + a_k X^k Y^k \tag{4}$$

1.5. Formula

$$Jac(XY,f^k) = kf^{k-1}Jac(XY,f) \tag{5}$$

1.6. Formula

$$Jac(XY,f\cdot g) = fJac(XY,g) + gJac(XY,f) \tag{6}$$

1.7. Formula

$$Jac(XY,g_1^k) = 0 \Leftrightarrow g_1 = 0, k \geq 1 \\ kg_1^{k-1}Jac(XY,g_1) = 0 \Rightarrow g_1 = 0 \text{or} Jac(XY,g_1) = 0 \Rightarrow g_1 = 0 \tag{7}$$

2. The simplest examples

2.1. Example 1

To study the simplest case, we must assume that polynomials $f = X^2Y^2 + f_3 + f_2 + + f_1$ and $h = XY + h_1$ have constant jacobian $Jac(f,h) = Jac(f_1,h_1)$.
Then in sequence

1) $Jac(X^2Y^2, h_1) + Jac(f_3, XY) = 0$

therefore

$2XYJac(XY,h_1) = Jac(XY, f_3)$

so

$Jac(XY, 2XYh_1) = Jac(XY, f_3)$

and then

$f_3 = 2XYh_1$

2) $Jac(f_3, h_1) + Jac(f_2, XY) = 0$

therefore

$Jac(2XYh_1, h_1) = Jac(XY, f_2)$

so

$Jac(XY, h_1^2) = Jac(XY, f_2)$

and then

$f_2 = h_1^2 + a_1 XY$

3) $Jac(f_2, h_1) + Jac(f_1, XY) = 0$

therefore

$Jac(h_1^2 + a_1 XY, h_1) = Jac(XY, f_1)$

so

$Jac(XY, a_1 h_1) = Jac(XY, f_1)$

and then

$f_1 = a_1 h_1$

This means that $Jac(f_1,h_1) = 0$, the simplest polynomial having two zeros at infinity cannot have a constant nonzero jacobian (if it has a constant jacobian then the jacobian equals zero).
We may notice that in this elementary example
$f = h + a_1 h$
$f = X^2Y^2 + 2XYh_1 + (h_1^2 + a_1XY) + a_1h_1 = (XY + h_1)^2 + a_1(XY + h_1)$
This solution confirms the fact that jacobian $Jac(f,h) = 0$ (polynomials f and h are algebraically dependent in this trivial case).

2.2. Example 2

In the next example we may suppose that polynomials $f = X^2Y^2 + f_5 + f_4 + f_3 + + f_2 + f_1$ and $h = X^2Y^2 + h_3 + h_2 + h_1$ have constant jacobian. For the reader it might be difficult so let us try to move the pattern from example 1. In sequence

1) $Jac(X^3Y^3, h_3) + Jac(f_5, X^2Y^2) = 0$

therefore

$3X^2Y^2 Jac(XY, h_3) = 2XYJac(XY, f_5)$

so

$3XYJac(XY, h_3) = 2Jac(XY, f_5)$

and then

$f_5 = \frac{3}{2} XYh_3$

2) $Jac(X^3Y^3, h_2) + Jac(f_5, h_3) + Jac(f_4, X^2Y^2) = 0$

therefore

$$3X^2Y^2 Jac(XY, h_2) + Jac(\frac{3}{2}XYh_3, h_3) = 2XYJac(XY, f_4)$$

so

$$Jac(XY, 3X^2Y^2h_2) + Jac(XY, \frac{3}{4}h_3^2) = Jac(XY, 2XYf_4)$$

and then

$$2XYf_4 = 3X^2Y^2h_2 + \frac{3}{4}h_3^2 + a_3X^3Y^3$$

it means that XY divides h_3^2, so it divides also h_3 so

$h_3 = XYh_{3/1}$

therefore

$$2XYf_4 = 3X^2Y^2h_2 + \frac{3}{4}X^2Y^2h_{3/1}^2 + a_3X^3Y^3$$

and then

$$f_4 = \frac{3}{2}XYh_2 + \frac{3}{8}XYh_{3/1}^2 + \frac{1}{2}a_3X^2Y^2$$

3) $Jac(X^3Y^3, h_1) + \underbrace{Jac(f_5, h_2)}_{\textcircled{1}} + \underbrace{Jac(f_4, h_3)}_{\textcircled{2}} + Jac(f_3, X^2Y^2) = 0$

$\textcircled{1}$ =

$$Jac(f_5, h_2) = Jac(\frac{3}{2}X^2Y^2h_{3/1}, h_2) = 3XYh_{3/1}Jac(XY, h_2) + \frac{3}{2}X^2Y^2Jac(h_{3/1}, h_2)$$

$\textcircled{2}$ =

$$Jac(f_4, h_3) = Jac(\frac{3}{2}XYh_2 + \frac{3}{8}XYh_{3/1}^2 + \frac{1}{2}a_3X^2Y^2, XYh_{3/1}) =$$

$$= \frac{3}{2}Jac(XYh_2, XYh_{3/1}) + \frac{3}{8}Jac(XYh_{3/1}^2, XYh_{3/1}) + \frac{1}{2}a_3Jac(X^2Y^2, XYh_{3/1}) =$$

$$= \frac{3}{2}XYh_2Jac(XY, h_{3/1}) - \frac{3}{2}XYh_{3/1}Jac(XY, h_2) + \frac{3}{2}X^2Y^2Jac(h_2, h_{3/1}) -$$

$$-\frac{3}{8}XYh_{3/1}^2Jac(XY, h_{3/1}) + a_3XYJac(h_2, h_{3/1})$$

therefore

$$3X^2Y^2Jac(XY, h_1) + \frac{3}{2}XYh_{3/1}Jac(XY, h_2) + \frac{3}{2}XYh_2Jac(XY, h_{3/1}) -$$

$$-\frac{3}{8}XYh_{3/1}^2Jac(XY, h_{3/1}) + a_3X^2Y^2Jac(XY, h_{3/1}) = 2XYJac(XY, f_3)$$

so

$$Jac(XY,3XYh_1)+Jac(XY,\frac{3}{2}h_{3/1}h_2)-\frac{1}{8}Jac(XY,h_{3/1}{}^3)+Jac(XY,a_3XYh_{3/1})=$$

$$=Jac(XY,2f_3)$$

and then

$$f_3=\frac{3}{2}XYh_1+\frac{3}{4}h_{3/1}h_2-\frac{1}{16}h_{3/1}{}^3+\frac{1}{2}a_3XYh_{3/1}$$

4) $Jac(f_5,h_1)+Jac(f_4,h_2)+Jac(f_3,h_3)+Jac(f_2,X^2Y^2)=0$

By performing tedious calculations we obtain

$$f_2=\frac{3}{4}h_{3/1}h_1+\frac{1}{2}a_3h_2+\frac{1}{2}a_2XY+\frac{3}{8}b_1^2XY$$

while

$$h_2=\frac{1}{4}h_{3/1}{}^2+b_1XY$$

5) $Jac(f_4,h_1)+Jac(f_3,h_2)+Jac(f_2,h_3)+Jac(f_1,X^2Y^2)=0$

By performing tedious calculations we obtain

$$f_1=\alpha_1h_1+\beta_1h_{3/1}$$

6) $Jac(f_3,h_1)+Jac(f_2,h_2)+Jac(f_1,h_3)=0$

By performing tedious calculations we obtain

$$h_1=\frac{1}{2}B_1h_{3/1}$$

Therefore we have received

$h_1=\frac{1}{2}B_1h_{3/1}$, $h_2=\frac{1}{4}h_{3/1}{}^2+B_1XY$ and $h_3=XYh_{3/1}$, which means, that

$$h=X^2Y^2+XYh_{3/1}+\frac{1}{4}h_{3/1}{}^2+B_1XY+\frac{1}{2}B_1h_{3/1}=(XY+\frac{1}{2}h_{3/1})^2+B_1(XY+\frac{1}{2}h_{3/1})$$

Also consequently

$$f=(X^3Y^3+\frac{1}{2}h_{3/1})^3+A_1(XY+\frac{1}{2}h_{3/1})^2+A_2(XY+\frac{1}{2}h_{3/1})$$

By selecting right parameters $A_1=\frac{1}{2}a_3$ and

$A_2=\frac{1}{2}a_2+\frac{3}{2}b_1XY+\frac{1}{2}a_3b_1+\frac{3}{8}b_1{}^2+\frac{3}{4}b_1h_{3/1}$ this means that

$$Jac(f_1,h_1)=Jac(\frac{1}{2}A_1h_{3/1},\frac{1}{2}B_1h_{3/1})=0$$

These examples show the general method.

3. Proposition

Let $f = X^k Y^k + f_{2k-1} + ... + f_1$, and $h = X^{k-1} Y^{k-1} + h_{2k-3} + ... + h_1$, $k \geq 2$

If $Jac(f, h) = Jac(f_1, h_1)$, then $Jac(f_1, h_1) = 0$

The idea of the proof. The particular parts of the proof lead to the following forms of polynomials *f* and *h*

$$f = (XY + \frac{1}{k-1} h_{2k-3/1})^k + A_1 (XY + \frac{1}{k-1} h_{2k-3/1})^{k-1} + ... + A_{k-1} (XY + \frac{1}{k-1} h_{2k-3/1})$$

And

$$h = (XY + \frac{1}{k-1} h_{2k-3/1})^{k-1} + B_1 (XY + \frac{1}{k-1} h_{2k-3/1})^{k-2} + ... + B_{k-2} (XY + \frac{1}{k-1} h_{2k-3/1})$$

While

$$h_{2k-3} = X^{k-2} Y^{k-2} h_{2k-3/1}$$

This means that polynomials *f* and *h* are algebraically dependent and it confirms that

$$Jac(f, h) = 0$$

Conclusions

We have received the whole set of polynomials formulas having two zeros at infinity in which the constant jacobian must vanish. We suppose that this is true for every polynomial mapping of two zeros at infinity. This note will be the aim of our next article.

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