Journal of Applied Mathematics and Computational Mechanics 2013, 12(3), 123-133

ON STRONG AUTOMORPHISMS OF DIRECT PRODUCTS OF WITT RINGS (I)

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Abstract. The notion of Witt ring is fundamental in bilinear algebra. Automorphisms of Witt rings have been investigated until recent years. In this paper we consider Witt rings which are direct products of finite number of other Witt rings. We shall present a necessary condition in order to group of all strong automorphisms of direct product of Witt rings be a direct product of groups of strong automorphisms of Witt rings which are factors in the direct product. Subsequently, there are considered some examples of Witt rings, where described condition is fulfilled.

Keywords: *Witt rings, quaternionic structures, the strong automorphisms*

Introduction

A fundamental notion in the algebraic theory of quadratic forms is the ring introduced in [1], called now the Witt ring of quadratic forms. The structure and properties of Witt ring $W(K)$ of quadratic forms over the field K depend strongly on the field K of coefficients of forms. In particular there is an essential difference between Witt rings over the fields of characteristic $\neq 2$ and Witt rings of the field of characteristic = 2.

One of the interesting problems about Witt rings is the description of their automorphisms. However, the task is difficult because of mentioned various structures of Witt rings. Therefore, there is no formula describing all automorphisms of all Witt rings. The descriptions of automorphisms of Witt rings known in the literature apply to separate classes of Witt rings (see for example [2-4]).

In this paper we consider abstract Witt rings introduced by M. Marshall in [5] as an abstract equivalent of well-known Witt rings of quadratic forms, which have the same algebraic properties as the original objects. We use well-known one-to-one correspondence between Witt rings and quaternionic structures in order to search for strong automorphisms of direct product of Witt rings. Let W be a Witt ring which is a direct product of finite number of Witt rings W_i, $1 \leq i \leq n$. As the main result, we present a necessary condition in order to group of all strong auto-

morphisms of direct product of Witt rings to be a direct product of groups of strong automorphisms of Witt rings that are factors in the direct product, i.e.

$$Aut(W) \cong \prod_{i=1}^{n} Aut(W_i)$$

Finally, we present some examples of Witt rings which are direct products of Witt rings where the above formula is true.

1. Preliminaries

1.1. Witt rings, quaternionic structures and their automorphisms

Following Marshall (cf. [5]) a *Witt ring* is said to be a pair $W = (R, G)$, where R is a commutative ring with unity 1 and G is a subgroup of the multiplicative group R^* which has exponent 2 and contains distinguished element -1 (where, as usual in a ring, $-r$ denotes the additive inverse of r). Furthermore, the following axioms hold:
$\mathcal{W}_1$: G generates R additively.
$\mathcal{W}_2$: The following Arason-Pfister property holds for $k = 1$ and $k = 2$:
If $r = a_1 + a_2 + \cdots + a_n \in I^k$, where I denotes the fundamental ideal of R generated by elements $r = a + b$, $a, b \in G$, $n < 2^k$, then $r = 0$.
$\mathcal{W}_3$: If $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$ and $n \geq 3$, then there exist $a, b, c_3, \ldots, c_n \in G$ such that $a_2 + \cdots + a_n = a + c_3 + \cdots + c_n$, $a_1 + a = b_1 + b$ (and, hence, $b_2 + \cdots + b_n = b + c_3 + \cdots + c_n$).

We will say that φ is a (*strong*) *isomorphism* of Witt rings $W_1 = (R_1, G_1)$ and $W_2 = (R_2, G_2)$ if $\varphi: R_1 \to R_2$ is a ring isomorphism such that $\varphi(G_1) = G_2$. A *strong automorphism of Witt ring* W is just isomorphism of W onto itself.

A useful tool for searching of automorphisms of Witt rings is a notion of quaternionic structure. Let G be a group of exponent 2, i.e. $a^2 = 1$ for all $a \in G$ with distinguished element $-1 \in G$ and let us denote $-a = -1 \cdot a$. Let Q be the set with distinguished element θ and let $q: G \times G \to Q$ be a surjective map. The triplet (G, Q, q) is called a *quaternionic structure,* if for every $a, b, c, d \in G$ the map q fulfills:
Q_1: $q(a, b) = q(b, a)$
Q_2: $q(a, -a) = \theta$
Q_3: $q(a, b) = q(a, c) \Rightarrow q(a, bc) = \theta$
Q_4: If $q(a, b) = q(c, d)$, then there exists such $x \in G$ that $q(a, b) = q(a, x)$ and $q(c, d) = q(c, x)$.
Two quaternionic structures (G_1, Q_1, q_1) and (G_2, Q_2, q_2) are *isomorphic* if there exists a group isomorphism $\sigma: G_1 \to G_2$ such that $\sigma(-1_1) = -1_2$ and $q_1(a, b) = \theta_1 \Leftrightarrow q_2\big(\sigma(a), \sigma(b)\big) = \theta_2$ for all $a, b \in G_1$. By *automorphism of a quaternionic structure* (G, Q, q) we understand any isomorphism $\sigma: (G, Q, q) \to (G, Q, q)$.

According to [5, Theorem 4.5] the category of Witt rings and the category of quaternionic structures are naturally equivalent. In particular for every Witt ring $W = (R, G)$ there exists a quaternionic structure (G, Q, q) associated to it and conversely for given quaternionic structure (G, Q, q) one can construct related Witt ring $W = (R, G)$. This fact makes it possible to use quaternionic structures in order to study properties of Witt rings when it is convenient. It was shown in [6] that strong automorphisms of Witt ring $W = (R, G)$ and automorphisms of quaternionic structure (G, Q, q) associated to W are in one-to-one correspondence and suitable groups of automorphisms are isomorphic, i.e. $Aut(W) \cong Aut(G, Q, q)$.

Let (G, Q, q) be a quaternionic structure. A (*quadratic*) *form of dimension* $n \geq 1$ over G is *n*-tuple $f = (a_1, \dots, a_n)$, where $a_1, \dots, a_n \in G$. A form f of dimension 2 is called *binary form*. Two forms of dimension n are called *equivalent* (or *isometric*) if:

(1) $n = 1$, $(a) \cong (b) \Leftrightarrow a = b$

(2) $n = 2$, $(a, b) \cong (c, d) \Leftrightarrow ab = cd$ and $q(a, b) = q(c, d)$

(3) $n > 2$, $(a_1, \dots, a_n) \cong (b_1, \dots, b_n) \Leftrightarrow \exists a, b, c_3, \dots, c_n \in G$ such that $(a_2, \dots, a_n) \cong (a, c_3, \dots, c_n)$, $(a_1, a) \cong (b_1, b)$ and $(b_2, \dots, b_n) \cong \cong (b, c_3, \dots, c_n)$.

The form $(1, a_1) \otimes \cdots \otimes (1, a_n)$, where $a_1, \dots, a_n \in G$, $n > 0$ is called *n-fold Pfister form*. We say that form f *represents element* $a \in G$ if there exist $a_2, \dots, a_n \in G$, such that $f \cong (a, a_2, \dots a_n)$. We denote the set of all elements represented by form f (*value set of the form* f) by $D(f)$. We have $f \cong g \Rightarrow D(f) = D(g)$. We shall often use the following formula proved by M. Marshall ([5, p. 74])

$$b \in D(1, -a) \Leftrightarrow q(a, b) = \theta \tag{1.1}$$

The above formula gives us the new tool for searching automorphisms of quaternionic structures. We can convert the second condition of definition of automorphisms of quaternionic structures $q(a, b) = \theta \Leftrightarrow q(\sigma(a), \sigma(b)) = \theta$ by $\sigma(D(1, a) = D(1, \sigma(a))$ for all $a \in G$ and use it when it is convenient.

Let $W = (R, G)$ be a Witt ring and let (G, Q, q) be the quaternionic structure associated to it. Then two forms $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$ are equivalent if $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_m$ in R and $m = n$. In many situations it is more convenient to use forms instead of elements of ring R.

1.2. Direct products and group rings

Let (G_k, Q_k, q_k), $1 \leq k \leq n$ be quaternionic structures such that $-1_k \in G_k$, $\theta_k \in G_k$. Let us accept the following notation: $G \coloneqq G_1 \times \cdots \times G_k$, $Q \coloneqq Q_1 \times \cdots \times Q\_k$, $-1 \coloneqq -1_1 \times \cdots \times -1_k$, $\theta \coloneqq \theta_1 \times \cdots \times \theta_k$ and let $q: G \times G \to Q$ be defined by $q([a_1, \dots, a_n], [b_1, \dots, b_n]) = [q_1(a_1, b_1), \dots, q_n(a_n, b_n)]$. Then the triplet (G, Q, q) is a quaternionic structure called the *product of quaternionic structures*

(G_k, Q_k, q_k), $1 \leq k \leq n$ (cf. [5], Chapter 5, §4) and denoted by $\prod_{k=1}^{n}(G_k, Q_k, q_k)$ or $(G_1, Q_1, q_1) \sqcap \cdots \sqcap (G_k, Q_k, q_n)$.

Moreover, using (1.1) we can write the value set of form $(1, a)$ for any $a \in G$ by $D(1, a) = D([1_1, \ldots, 1_n], [a_1, \ldots, a_n]) = D_1(1_1, a_1) \times \cdots \times D_n(1_n, a_n)$.

Let $(R_1, G_1), \ldots, (R_n, G_n)$ be Witt rings. Let R denote the subring of the ring $R_1 \times \cdots \times R_n$ generated additively by $G = G_1 \times \cdots \times G_n$. The pair $W = (R, G)$ is called a *direct product of Witt rings* $(R_1, G_1) \ldots (R_n, G_n)$ and denoted by $\prod_{i=1}^{n}(R_i, G_i)$ or $(R_1, G_1) \sqcap \cdots \sqcap (R_n, G_n)$.

Of course the quaternionic structure (G, Q, q) associated to the direct product $\prod_{k=1}^{n}(R_i, G_i)$ is isomorphic to the product $\prod_{k=1}^{n}(G_k, Q_k, q_k)$ of quaternionic structures associated to Witt rings (R_i, G_i).

Let $W' = (R', G')$ be a Witt ring. Let R denotes the group ring $R[\Delta]$ of the group Δ with coefficients in the ring R and let $G = \{ax: a \in G', x \in \Delta\}$. Then $W = (R, G)$ is a Witt ring called *group Witt ring* ([5, Proposition 5.16]). The group G should be denoted by $G'\Delta$ since it is a subset of $R'[\Delta]$. In order to make notation more clear we will use in the sequel notation $G' \times \Delta$ and an element ax we will denote by $[a, x]$.

According to [5], every element in the set $\boldsymbol{a} \in G \backslash (G' \times \{1_\Delta\})$ fulfills $D(1, \boldsymbol{a}) = \{1, \boldsymbol{a}\}$ ([5, Chapter 5, §8]). Therefore if $\boldsymbol{a} = [a, 1_\Delta] \in G' \times \Delta$, then $D(1, \boldsymbol{a}) = D'(1', a) \times \{1_\Delta\}$ and if $\boldsymbol{a} = [a, x] \in G' \times \Delta$, $x \neq 1_\Delta$, then $D(1, \boldsymbol{a}) = \{1, \boldsymbol{a}\}$.

1.3. Automorphisms of direct products of Witt rings and associated quaternionic structures

In this section we shall describe some conditions that allow one to find out for which Witt rings the group of strong automorphisms of their direct product equals to the direct product of groups of strong automorphisms of Witt rings being the factors.

Let us first recall a simple fact about automorphisms of quaternionic structures. Let $S(n)$ denotes the set of all permutations of n-element set, i.e. the set of all bijections of the set $\{1, \ldots, n\}$ onto itself.

Lemma 1.1. *Let* $(G, Q, q) \coloneqq \prod_{i=1}^{n}(\tilde{G}, \tilde{Q}, \tilde{q})$ *be a product of* n *copies of a quaternionic structure* $(\tilde{G}, \tilde{Q}, \tilde{q})$ (*n-th power of* $(\tilde{G}, \tilde{Q}, \tilde{q})$). *For every system of automorphisms* $\sigma_1, \ldots, \sigma_n \in Aut(\tilde{G}, \tilde{Q}, \tilde{q})$ *and for every permutation* $\alpha \in S(n)$ *a map* $\sigma: G \to G$ *defined by* $\sigma([a_1, \ldots, a_n]) \coloneqq [\sigma_1(a_{\alpha(1)}), \ldots, \sigma_n(a_{\alpha(n)})]$ *is an automorphism of quaternionic structure* (G, Q, q).

Proof. See [2, Proposition 2.1].■

Let $G = \prod_{k=1}^{n}(G_k, Q_k, q_k)$. Let us denote the subgroup $\{1\} \times \cdots \times \{1\} \times G_k \times \{1\} \times \cdots \times \{1\}$ of the group $G = G_1 \times \cdots \times G_n$ by G_k', where $1 \leq k \leq n$. We will

say that an automorphism of the group G *preserves the factors of the product* $\prod_{k=1}^{n}(G_k, Q_k, q_k)$ if for all $k \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\}$ such that $\sigma(G_k') = G_j'$.

Lemma 1.2. *Let* $(G,Q,q)^n$ *be n-th power of* (G,Q,q). *If any automorphism of the quaternionic structure* $(G,Q,q)^n$ *preserves the factors of the product* $(G,Q,q)^n$, *then*

$$Aut((G,Q,q)^n) \cong \big(Aut(G,Q,q)\big)^n \ltimes S(n).$$

Proof. Assume that $\sigma_1, \sigma_2, \dots, \sigma_n \in Aut((G,Q,q))$ and $\alpha \in S(n)$. Let $\sigma([a_1, \dots, a_n]) = [\sigma_1(a_{\alpha^{-1}(1)}), \dots, \sigma_n(a_{\alpha^{-1}(n)})]$ for all $[a_1, \dots, a_n] \in G^n$. By Lemma 2.1 σ is an automorphism of quaternionic structure $(G,Q,q)^n$. We define a map Φ: $\big(Aut(G,Q,q)\big)^n \ltimes S(n) \to Aut((G,Q,q)^n)$ by $\Phi\big(([\sigma_1, \dots, \sigma_n], \alpha)\big) \coloneqq \sigma$.

In order to prove that Φ is a group homomorphism we compare $\Phi\big(([\sigma_1, \dots, \sigma_n], \alpha) * ([\tau_1, \dots, \tau_n], \beta)\big)$ and $\Phi([\sigma_1, \dots, \sigma_n], \alpha) \circ \Phi([\tau_1, \dots, \tau_n], \beta)$ for all $[a_1, \dots, a_n] \in G^n$. By definition of multiplication in semi-direct product of groups we get

$\Phi\big([\sigma_1 \circ \tau_{\alpha^{-1}(1)}, \dots, \sigma_n \circ \tau_{\alpha^{-1}(n)}], \alpha \circ \beta\big)[a_1, \dots, a_n] =$

$= [\sigma_1 \circ \tau_{\alpha^{-1}(1)}(a_{(\alpha\circ\beta)^{-1}(1)}), \dots, \sigma_n \circ \tau_{\alpha^{-1}(n)}(a_{(\alpha\circ\beta)^{-1}(n)})].$

On the other hand

$\Phi([\sigma_1, \dots, \sigma_n], \alpha) \circ \Phi([\tau_1, \dots, \tau_n], \beta)[a_1, \dots, a_n]=$

$= \Phi([\sigma_1, \dots, \sigma_n], \alpha)[\tau_1(a_{\beta^{-1}(1)}), \dots, \tau_n(a_{\beta^{-1}(n)})] =$

$= \left[\sigma_1\left(\tau_{\alpha^{-1}(1)}\left(a_{\beta^{-1}\left(\alpha^{-1}(1)\right)}\right)\right), \dots, \sigma_n\left(\tau_{\alpha^{-1}(n)}\left(a_{\beta^{-1}\left(\alpha^{-1}(n)\right)}\right)\right)\right] =$

$= [\sigma_1 \circ \tau_{\alpha^{-1}(1)}(a_{(\alpha\circ\beta)^{-1}(1)}), \dots, \sigma_n \circ \tau_{\alpha^{-1}(n)}(a_{(\alpha\circ\beta)^{-1}(n)})]$

as in previous calculation. It proves that Φ is a group homomorphism.

By hypothesis a permutation $\alpha \in S(n)$ determines a map $\sigma_{\alpha(i)}: G_i \to G_{\alpha(i)}$, $i = 1, \dots, n$. Let $\pi_i: G^n \to G$ be the map such that $\pi_i([a_1, \dots a_n]) = a_i$ and let $\iota_i: G \to G^n$ be the map such that $\iota_i(a) = [1, \dots, a, \dots, 1]$, where a is on i-th position. Then it is easy to show that $\pi_{\alpha(i)} \circ \sigma_{\alpha(i)} \circ \iota_i: G \to G$ is an automorphism of quaternionic structure (G,Q,q). With above notation we have $\sigma([a_1, \dots, a_n]) =$ $= [\sigma_1(a_{\alpha^{-1}(1)}), \dots, \sigma_n(a_{\alpha^{-1}(n)})]$ for all $[a_1, \dots, a_n] \in G_1 \times \cdots \times G_n$, $i = 1, \dots n$. It follows that Φ is a surjection.

Assume that $([\sigma_1, \dots, \sigma_n], \alpha) \in \big(Aut(G,Q,q)\big)^n$ and that $\Phi([\sigma_1, \dots, \sigma_n], \alpha)$ is identity. Then for all $[a_1, \dots, a_n] \in G^n$ we have

$$\Phi([\sigma_1, \dots, \sigma_n], \alpha)[a_1, \dots, a_n] = [a_1, \dots, a_n] \tag{1.2}$$

Suppose that α is not identity permutation, hence there exists $i \in \{1, \dots n\}$ such that $\alpha^{-1}(i) \neq i$. Let $\alpha^{-1}(i) = j$. Let us consider a sequence $[a_1, \dots, a_n]$ such that

$a_j \neq 1$ and $a_l = 1$ for all other indices $l \in \{1, \dots n\}$. By (2.2) we get $\sigma_i\left(a_{\alpha^{-1}(i)}\right) = a_i$ for all $1 \leq i \leq n$. Thus $\sigma_i\left(a_j\right) = a_i = 1$ since $i \neq j$. That contradicts to the choice of element $[a_1, \dots, a_n]$. That means α must be identity permutation. Thus $\sigma_i(a_i) = a_i$ for all $1 \leq i \leq n$, hence σ is the identity map and it follows that Φ is injection.

That ends the proof that Φ is an isomorphism of the groups $Aut((G, Q, q)^n)$ and $\left(Aut(G, Q, q)\right)^n \ltimes S(n)$. ■

Now we can study more generally a product of quaternionic structures fulfilling properties described in Lemma 1.2 and its group of automorphisms.

Theorem 1.3. *Let $\mathcal{S} = \{(G_1, Q_1, q_1), \dots (G_n, Q_n, q_n)\}$ be a set of quaternionic structures such that every automorphism of the quaternionic structure $\prod_{i=1}^{n}(G_i, Q_i, q_i)$ preserves the factors of the product. Let $\mathcal{C}_1, \dots \mathcal{C}_m$ of cardinality $k_1, \dots, k_m$, respectively, be the classes of partition of the set $\mathcal{S}$ with respect to isomorphism of quaternionic structures and assume (without loss of generality) that (G_i, Q_i, q_i) are representatives of classes $\mathcal{C}_i$ for all $i = 1, \dots, m$. Then*

$$Aut\left(\prod_{(G,Q,q)\in\mathcal{S}} (G, Q, q)\right) \cong \prod_{i=1}^{m}\left(\left(Aut(G_i, Q_i, q_i)\right)^{k_i} \ltimes S(k_i)\right)$$

Proof. If σ is an automorphism of the quaternionic structure $\prod_{(G,Q,q)\in\mathcal{S}}(G, Q, q)$, then by hypothesis of Lemma 1.2 for every $i = 1, \dots, m$ there exists $j \in \{1, \dots, n\}$ such that $\sigma(G_i') = G_j'$ and $(G_i, Q_i, q_i) \in \mathcal{C}_i$. It follows that

$$Aut\left(\prod_{(G,Q,q)\in\mathcal{S}} (G, Q, q)\right) \cong \prod_{i=1}^{m} Aut\left(\prod_{(G,Q,q)\in\mathcal{C}_i} (G, Q, q)\right)$$

Now by the previous theorem we get

$$Aut\left(\prod_{(G,Q,q)\in\mathcal{C}_i} (G, Q, q)\right) \cong \left(Aut(G_i, Q_i, q_i)\right)^{k_i} \ltimes S(k_i)$$

which finishes the proof. ■

The following corollary is a direct consequence of theorem 1.3.

Theorem 1.4. *Let (G, Q, q) be a finite product of pairwise non-isomorphic quaternionic structures $(G_1, Q_1, q_1), \dots, (G_n, Q_n, q_n)$ such that every automorphism of the quaternionic structure $\prod_{i=1}^{n}(G_i, Q_i, q_i)$ preserves the factors of the product. Then*

$$Aut(G,Q,q) \cong \prod_{i=1}^{n} Aut(G_i,Q_i,q_i).$$

One can translate the expressions in Theorems 1.3 and 1.4 to the language of Witt rings.

2. Strong automorphisms of direct products of group Witt rings

2.1. Witt rings of local types

Our first application of the results of previous section concerns Witt rings of local type. Recall that a quaternionic structure (G,Q,q) is said to be of *local type* if G is finite and $|D(1,a)| = \frac{1}{2}|G|$ for all $-1 \neq a \in G$ (and, as always in quaternionic structures, $D(1,-1) = G$). The Witt ring (R,G) associated to (G,Q,q) of local type is called *Witt ring of local type*. By [2, Lemma 2.2], every automorphism of finite product of quaternionic structures of local type preserves factors of the product. Therefore if W is a Witt ring of local type which is a direct product of Witt rings of local type $W_1, \dots, W_n$ then by Theorem 1.4 we conclude

$$Aut(W) \cong \prod_{i=1}^{n} Aut(W_i)$$

if Witt rings $W_1, \dots, W_n$ are pairwise non-isomorphic (compare [2, Corollary 2.6]) and by Theorem 1.3 we get

$$Aut(W) \cong \prod_{i=1}^{n} Aut(W_i) \ltimes S(n)$$

if Witt rings $W_1, \dots, W_n$ can be divided into classes of Witt rings with respect to strong isomorphism (compare [2, Theorem 2.4]). In fact the results in [2] are special cases of our Theorems 1.3 and 1.4.

2.1. Group Witt rings

Since our next example involves Witt rings which are group rings with coefficients in Witt rings of local type first we recall the structure of some Witt rings of local type, their associated quaternionic structures and value sets of binary Pfister forms $(1,a)$, $a \in G$.

Example 2.1.

1) Let $W(\mathbb{Q}_3)$ be the Witt ring of local type realized by 3-adic field $\mathbb{Q}_3$. The ring $W(\mathbb{Q}_3)$ is isomorphic to abstract Witt ring $\mathbb{Z}/4\mathbb{Z}[C_2]$ - the group ring of the two--element multiplicative cyclic group $C_2 = \{1,x\}$ with coefficients in the ring $\mathbb{Z}$

of integers ([5]). The associated quaternionic structure $\left(G_{\mathbb{Q}_3}, Q_{\mathbb{Q}_3}, q_{\mathbb{Q}_3}\right)$ is based on the group $G_{\mathbb{Q}_3} = \{1, -1, p, -p\}$ where $p = 3$ (compare [7, Theorem 2.2, p. 152] or [8, Corollary at p. 18]). Therefore the value sets of 1-fold Pfister forms are:
$D(1,1) = \{1, -1\}$, $D(1, -1) = G_{\mathbb{Q}_3}$, $D(1, p) = \{1, p\}$, $D(1, -p) = \{1, -p\}$.

It is easy to calculate that the quaternionic structure $\left(G_{\mathbb{Q}_3}, Q_{\mathbb{Q}_3}, q_{\mathbb{Q}_3}\right)$ (and consequently Witt ring $W(\mathbb{Q}_3)$) has two automorphisms: σ_1 which is identity and σ_2 such that $\sigma_2(p) = -p$. One can describe the group $Aut\left(\left(G_{\mathbb{Q}_3}, Q_{\mathbb{Q}_3}, q_{\mathbb{Q}_3}\right)\right)$ in another way with use of the results presented in [3] to the group Witt ring $\mathbb{Z}/4\mathbb{Z}[C_2]$.

2) Let $W(\mathbb{Q}_5)$ be the Witt ring of local type realized by 5-adic field $\mathbb{Q}_5$. The ring $W(\mathbb{Q}_5)$ is isomorphic to abstract Witt ring $\mathbb{Z}/2\mathbb{Z}[C_4]$ - the group ring of the 4-element group $\{1, x, y, xy\}$ of exponent 2 with coefficients in the ring $\mathbb{Z}$ [5]. The group $G_{\mathbb{Q}_5}$ in quaternionic structure $\left(G_{\mathbb{Q}_5}, Q_{\mathbb{Q}_5}, q_{\mathbb{Q}_5}\right)$ associated to $W(\mathbb{Q}_5)$ can be written as $G_{\mathbb{Q}_5} = \{1, p, u, up\}$, where $\left(\frac{u}{p}\right) = -1$ and $p = 5$ (for example $u = 2$) (compare [7, Theorem 2.2, p. 152] or [8, Corollary at p. 18]). Therefore, the value sets of 1-fold Pfister forms are:
$D(1,1) = G_{\mathbb{Q}_5}$, $D(1, p) = \{1, p\}$, $D(1, u) = \{1, u\}$, $D(1, up) = \{1, up\}$.

One can calculate that in this case the group of strong automorphisms $Aut(W(\mathbb{Q}_5))$ has 6 elements (Compare also [3, Theorem 2.2] and use it to the group Witt ring $\mathbb{Z}/2\mathbb{Z}[C_4]$).

Example 2.2.

Consider Witt ring $W = (R, G)$ which is a direct product of two group Witt rings, namely $W \cong W_1 \sqcap W_2 = (W(\mathbb{Q}_3) \sqcap W(\mathbb{Q}_3))[C_2] \sqcap (W(\mathbb{Q}_3))[C_2]$.
Let us write out the quaternionic structure associated to the Witt ring W.
Using usual calculation in group Witt rings (see [5, Chapter 5, §4] and [3]) and our notation concerning group Witt rings we get $G \cong G_1 \times G_2 = \left(G_{\mathbb{Q}_3} \times G_{\mathbb{Q}_3} \times C_2\right) \times \times (G_{\mathbb{Q}_3} \times C_2)$. Then $|G| = |G_1| \cdot |G_2| = (4 \cdot 4 \cdot 2) \cdot (4 \cdot 2) = 256$.

Since we will use the cardinality of value sets of binary Pfister forms in W_1 and W_2 (and consequently in W), then we will describe it precisely.

The ring W_1 is the group ring of the group $C_2 = \{1, x\}$ with coefficients in direct product of Witt rings of local type $W(\mathbb{Q}_3) \sqcap W(\mathbb{Q}_3)$. Using information about value sets of Witt rings of local type and about the way of calculation of value sets in group Witt rings [compare [3]) we can calculate value sets in W_1 as follows:

1) $|D_1(1_1, -1_1)| = 32$.
2) There are 16 elements such that $|D_1(1_1, d)| = 2$, where $\pm 1_1 \neq d \in G_1$ is of the form $d = [b, x]$, $x \neq 1$, $b \in G_{\mathbb{Q}_3} \times G_{\mathbb{Q}_3}$, $x \in C_2$ (then $D_1(1_1, d) = \{1_1, d\}$).
3) There are 9 elements of the form $-1_1 \neq d \neq [b, 1] \in (G_{\mathbb{Q}_3} \times G_{\mathbb{Q}_3}) \times C_2$, such that $|D_1(1_1, d)| = 4$ (in particular $|D_1(1_1, 1_1)| = 4$).

4) There are 6 elements of the form $\pm 1_1 \neq d = [b, 1] \in (G_{\mathbb{Q}_3} \times G_{\mathbb{Q}_3}) \times C_2$, such that $|D_1(1_1, d)| = 8$.

The ring W_2 is the group ring of the group C_2 with coefficients in Witt ring of local type $W(\mathbb{Q}_3)$. Therefore W_2 fulfills the following conditions:
1) $D_2(1_2, -1_2) = G_2$,
2) $D_2(1_2, 1_2)| = \{1_2, -1_2\}$,
3) $D_2(1_2, c)| = \{1_2, c\}$ for all $c \in G_2$, $c \neq \pm 1_2$.

Since the group G has cardinality 256 then the task of searching of all its automorphisms is very difficult. In order to calculate the number of all strong automorphisms of Witt ring W we used a computer program, where the group G is considered as a vector space over the two-element field $\mathbb{F}_2$. The algorithm and full description of the program one can find in [9]. With use of this tool we got the following result: $|Aut(W)| = 3072$.

Now we shall prove that any automorphism of the quaternionic structure $(G_1 \times G_2, Q_1 \times Q_2, q_1 \times q_2)$ preserves the factors of the product. This fact allows us to apply our theorem 1.4 which implies that $Aut(W) \cong Aut(W_1) \times Aut(W_2)$ and consequently $|Aut(W)| = |Aut(W_1)| \cdot |Aut(W_2)| = 128 \cdot 24 = 3072$.

Let $(G, Q, q) = (G_1 \times G_2, Q_1 \times Q_2, q_1 \times q_2)$ be the quaternionic structure associated to above Witt ring $W \cong W_1 \sqcap W_2$. We shall show that for any $\sigma \in Aut(G, Q, q)$ the following conditions hold:
1) $\sigma(G_1 \times \{1_2\}) = G_1 \times \{1_2\}$ and
2) $\sigma(\{1_1\} \times G_2) = \{1_1\} \times G_2$.

The proof is based on knowledge about value sets of 1-fold Pfister forms in quaternionic structures (G_1, Q_1, q_1) and (G_2, Q_2, q_2).

Let σ be fixed automorphism of (G, Q, q).
Step 1. Consider an element $\boldsymbol{a} = [1_1, -1_2] \in G = G_1 \times G_2$. We know that $|D(\mathbf{1}, \boldsymbol{a})| = |D([1_1, 1_2], [-1_1, -1_2])| = |D_1(1_1, 1_1) \times D_2(1_1, -1_2)| = 4 \cdot 8 = 32$. Assume that $\sigma(\boldsymbol{a}) = [x, y]$. Since σ preserves value sets of forms (as automorphism of quaternionic structure), hence in particular $|D([1_1, 1_2], [x, y])| = 32$. Suppose that $y \neq -1$ in G_2. Then $|D_2([1_2, y])| = 2$. Therefore, if $|D([1_1, 1_2], [x, y])| = |D_1(1_1, x)| \cdot |D_2([1_2, y])| = 32$, it follows $|D_1(1_1, x)| = 16$ in G_2, contradiction, since it does not hold for any $x \in G_1$. Thus $y = -1_2$ and $\sigma(a) = \sigma([1_1, -1_2]) = [x, -1_2]$ for some $x \in G_1$ such that $|D_1(1_1, x)| = 4$.

If we take the opposite element, then $\sigma(-\boldsymbol{a}) = \sigma([-1_1, 1_2]) = -\sigma(\boldsymbol{a}) = [-x, 1_2]$ for some $x \in G_1$. We have $|D(1_1, -\boldsymbol{a})| = |D([1_1, 1_2], [-1_1, 1_2])| = |D_1(1_1, -1_1)| \cdot |D_2(1_2, 1_2)| = 32 \cdot 2 = 64$, hence also $|D(1_1, \sigma(-\boldsymbol{a}))| = 64$. Now we calculate $64 = |D([1_1, 1_2], [-x, 1_2])| = |D_1(1_1, -x) \times D_2(1_2, 1_2)|$. Since $|D_2(1_2, 1_2)| = 2$, then $|D_1(1_1, -x)| = 32$ and it follows that $-x = -1_1$ and $x = 1_1$.

Finally, for any $\sigma \in Aut(G, Q, q)$ we have shown that $\sigma([1_1, -1_2]) = [1_1, -1_2]$ (and $\sigma([-1_1, 1_2]) = [-1_1, 1_2]$).

Step 2. Consider an element $\boldsymbol{a} = [-1_1, y] \in G = G_1 \times G_2$ and such that $y \neq \pm 1_2$. We have $|D(\mathbf{1}, \boldsymbol{a})| = |D([1_1, 1_2], [-1_1, y])| = |D_1(1_1, -1_1) \times D_2(1_2, y)| = 32 \cdot 2 = 64$. Assume that $\sigma(\boldsymbol{a}) = \sigma([-1_1, y]) = [x', y'] = \boldsymbol{a}'$ for some $x' \in G_1$, $y' \in G_2$. Since σ preserves value sets of forms, then $|D([1_1, 1_2], [x', y'])| = 64$. In our Witt ring either of the two cases occur:
a) $|D([1_1, 1_2], [x', y'])| = |D_1(1_1, x')| \cdot |D_2(1_2, y')| = 32 \cdot 2$ or
b) $|D([1_1, 1_2], [x', y'])| = |D_1(1_1, x')| \cdot |D_2(1_2, y')| = 8 \cdot 8$.

Suppose that the case b) holds. It is possible only if:
(i) $|D_1(1_1, x')| = 8$, hence we get $x' \neq \pm 1_1$ and
(ii) $|D_2(1_2, y')| = 8$, thus $y' = -1_2 \in G_2$.

Now we consider the opposite element. We have $\sigma(-\boldsymbol{a}) = \sigma([1_1, -y]) = = [-x', 1_2] = -\boldsymbol{a}'$. Since $y \neq \pm 1_2$, it follows that $|D(\mathbf{1}, -\boldsymbol{a})| = |\mathrm{D}([1_1, 1_2], [1_1, -\mathrm{y}])| = |\mathrm{D}_1(1_1, 1_1) \times \mathrm{D}_2(1_2, -\mathrm{y})| = |\mathrm{D}_1(1_1, 1_1)| \cdot |\mathrm{D}_2(1_2, -\mathrm{y})| = 4 \cdot 2 = 8$.
Next, since σ is an isomorphism of quaternionic structures (and preserves value sets of forms), we get $8 = \left|D\left(\mathbf{1}, \sigma(-\boldsymbol{a})\right)\right| = |D\left([1_1, 1_2]\right], \sigma([1_1, -y]))| = = |D([1_1, 1_2], [-x', 1_2])| = |D_1(1_1, -x')| \cdot |D_2(1_2, 1_2)| = |D_1(1_1, -x')| \cdot 2$. Thus $|D_1(1_1, -x')| = 4$. It follows that x' is an element of $G_1 = G_{\mathbb{Q}_3} \times G_{\mathbb{Q}_3} \times C_2$ such that $x' = [s, t, 1]$ for some $s, t \in G_{\mathbb{Q}_3}$. Notice first that $x' \neq [1, -1,1]$ and $x' \neq \neq [-1,1,1]$. In fact, suppose that $x' = [1, -1,1]$. Then $|D_1(1_1, x')| = |D_1([1,1,1], [1, -1,1])| = 8$, and consequently $|D_1(1_1, -x')| = = |D_1([1,1,1], [-1,1,1])| = 8$, a contradiction (because we have assumed $|D_1(1_1, -x')| = 4$). Analogously, $x' = [-1,1,1]$ is not possible, because it implies $|D_1(1_1, -x')| = |D_1([1,1,1], [1, -1,1])| = 8$ and we get the same contradiction. There are two cases possible:

1) $|D_{\mathbb{Q}_3}(1, s)| = 4$, thus $s = -1 \in G_{\mathbb{Q}_3}$ and $|D_{\mathbb{Q}_3}(1, t)| = 2$, hence $t \neq -1$.
   By previous notation we have $G_{\mathbb{Q}_3} = \{1, -1, p, -p\}$ where $p \neq \pm 1$ and $C_2 = \{1, x\}$, so we can write

$$x' = [-1, p, 1] \quad \text{or} \quad x' = [-1, -p, 1] \tag{2.1}$$

2) $|D_{\mathbb{Q}_3}(1, s)| = 2$, thus $s \neq -1$ and $|D_{\mathbb{Q}_3}(1, t)| = 4$, hence $t = -1$ and then

$$x' = [p, -1,1] \quad \text{or} \quad x' = [-p, -1,1] \tag{2.2}$$

since we have excluded the cases $x' = [1, -1,1]$ and $x' = [-1, -1,1]$.
Now we use results from Step 1. We know that the element $[-1_1, 1_2]$ is of the form $[-1_1, y]$, hence by previous calculation $[-1_1, 1_2] \in D([1_1, 1_2], [-1_1, y])$ and by Step 1 for any automorphism σ of quaternionic structure (G, Q, q) we have $\sigma([-1_1, 1_2]) = [-1_1, 1_2]$. Therefore using, again the properties of σ we get $[-1_1, 1_2] = \sigma([-1_1, 1_2]) \in D([1_1, 1_2], \sigma([-1_1, y]) = D([1_1, 1_2], [x', y']) = = D_1(1_1, x') \times D_2(1_2, y')$. It follows that $-1_1 \in D_1(1_1, x')$, therefore using twice

(1.1) we get $-x' \in D_1(1_1, 1_1) = \{[-1,-1,1],[1,1,1],[1,-1,1],[-1,1,1]\}$, what is a contradiction to (2.1), (2.2) and $|D_1(1_1, -x')| = 4$. Therefore (ii) $y' = -1_2 \in G_2$ is false and consequently the case b) does not occur. It follows only the case a) is true, that means for any $\sigma \in Aut(G, Q, q)$ we have $\sigma([-1_1, y]) = [-1_1, y']$ for some $y' \in G_2$, such that $|D_2(1_2, y')| = 2$. It follows that for any $\boldsymbol{a} = [1, y] \in$ $\in G_1 \times G_2$ holds $\sigma(-a) = \sigma([1_1, y]) = [1_1, y']$ for some $y' \in G_2$, what means that $\sigma(\{1_1\} \times G_2) = \{1_1\} \times G_2$.

Step 3. Notice that
$D([1_1, 1_2], [-1_1, 1_2]) \cap D([1_1, 1_2], [-1_1, y]) =$
$= \big(D_1(1_1, -1_1) \times D_2(1_2, 1_2)\big) \cap \big(D_1(1_1, -1_1) \times D_2(1_2, y)\big) =$
$= \big(D_1(1_1, -1_1) \cap D_1(1_1, -1_1)\big) \times \big(D_2(1_2, 1_2) \cap D_2(1_2, y)\big) = G_1 \times \{1_2\}.$
Therefore for any $\sigma \in Aut(G, Q, q)$ we get
$\sigma(G_1 \times \{1_2\}) =$
$= \sigma(D([1_1, 1_2], [-1_1, 1_2]) \cap D([1_1, 1_2], [-1_1, y])) =$
$= D([1_1, 1_2]), \sigma([-1_1, 1_2])) \cap D([1_1, 1_2], \sigma([-1_1, y])) =$
$= D([1_1, 1_2], [-1_1, 1_2]) \cap D([1_1, 1_2], [-1_1, y']) = G_1 \times \{1_2\}$
q.e.d.

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