# THE DETERMINANTS OF THE BLOCK BAND MATRICES BASED ON THE *n*-DIMENSIONAL FOURIER EQUATION PART 2

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**Abstract.** This work is a continuation of the considerations concerning the determinants of the band block matrices on the example of the *n*-dimensional Fourier equation (work Part 1). The discussion will concern the special case called the three-dimensional Fourier equation.

***Keywords:*** *block matrices, band matrices, determinant, Fourier equation*

## Introduction

The 3D Fourier equation can describe some of the transport phenomena which are typical of the irreversible processes occurring in nature. These phenomena depend on the transport energy, matter, momentum or electric charge on a macroscopic scale.

The process of transport at the time could be presented by a partial differential equation

$$\frac{\partial^2\Phi}{\partial x_1^2}+\frac{\partial^2\Phi}{\partial x_2^2}+\frac{\partial^2\Phi}{\partial x_3^2}=K\frac{\partial\Phi}{\partial t} \tag{1}$$

where $K$ is a constant characterizing the process.

This equation describes the propagation of a certain scalar quantity $\Phi\left(x_1,x_2,x_3,t\right)$.

Therefore, the Fourier equation appears in both the heat transfer and, for example, hydrogeology.

In hydrogeology the three-dimensional Fourier equation $\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+\frac{\partial^2 H}{\partial z^2}= =\frac{S}{T}\frac{\partial H}{\partial t}$, where $H$ - height of hydraulic, $S$ - water capacity, $T$ - conductivity and $t$ - time, describes the transient filtering in a homogeneous and isotropic domain without internal sources.

In this paper we consider the transient diffusion of heat in the 3D domain.

## 1. Solution of the problem

The Fourier equation describing the heat flow is in the form [1]

$$\lambda\left(\frac{\partial^2 T(x_1, x_2, x_3, t)}{\partial x_1^2}+\frac{\partial^2 T(x_1, x_2, x_3, t)}{\partial x_2^2}+\frac{\partial^2 T(x_1, x_2, x_3, t)}{\partial x_3^2}\right)= \\ = \rho c \frac{\partial T(x_1, x_2, x_3, t)}{\partial t} \tag{2}$$

where λ is a thermal conductivity, $c$ is a specific heat, $\rho$ is a mass density and $T$, $x_1$, $x_2$, $x_3$, $t$ denotes the temperature, geometrical co-ordinates and time.

Let us consider the following inner element $(\Delta x_1, \Delta x_2, \Delta x_3)$ and the internal nodes of the spatial grid (Fig. 1).

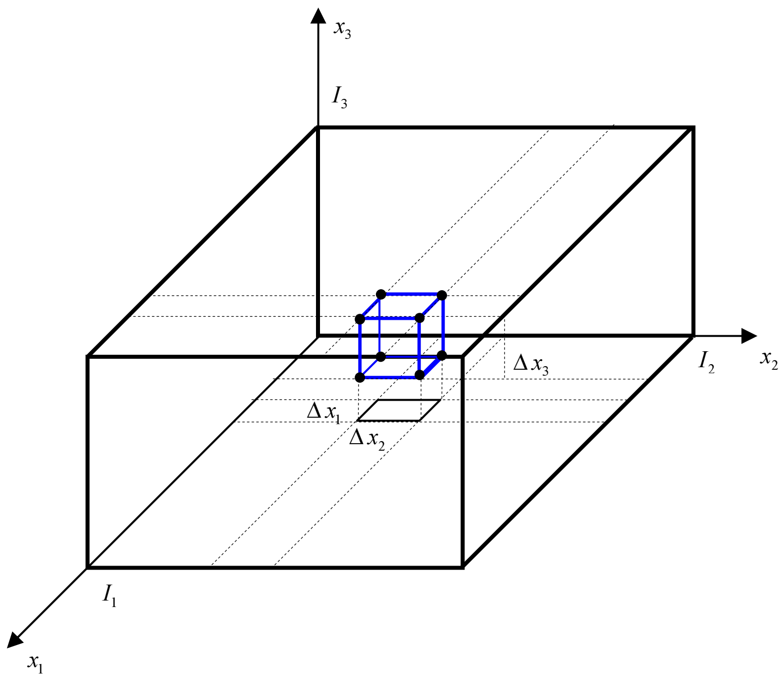


Fig. 1. An example of the inner element and the internal nodes of the spatial grid

The second differential spatial derivatives are as follows:

$$\begin{aligned}
\frac{\partial^2 T}{\partial x_1^2} &= \frac{T_{i_1-1,i_2,i_3,l} - 2T_{i_1,i_2,i_3,l} + T_{i_1+1,i_2,i_3,l}}{\left(\Delta x_1\right)^2}, \quad 1 \le i_1 \le m_1 - 1 \\
\frac{\partial^2 T}{\partial x_2^2} &= \frac{T_{i_1,i_2-1,i_3,l} - 2T_{i_1,i_2,i_3,l} + T_{i_1,i_2+1,i_3,l}}{\left(\Delta x_2\right)^2}, \quad 1 \le i_2 \le m_2 - 1 \\
\frac{\partial^2 T}{\partial x_3^2} &= \frac{T_{i_1,i_2,i_3-1,l} - 2T_{i_1,i_2,i_3,l} + T_{i_1,i_2,i_3+1,l}}{\left(\Delta x_3\right)^2}, \quad 1 \le i_3 \le m_3 - 1
\end{aligned} \tag{3}$$

and the first differential derivative of the time has a form

$$\frac{\Delta T}{\Delta t} = \frac{T_{i_1,i_2,i_3,l} - T_{i_1,i_2,i_3,l-1}}{\Delta t}, \quad 1 \le l \le q \tag{4}$$

Therefore, the Finite Difference Method leads to the internal system of equations

$$\begin{aligned}
&\frac{\lambda}{\left(\Delta x_1\right)^2} T_{i_1-1,i_2,i_3,l} - \frac{2\lambda}{\left(\Delta x_1\right)^2} T_{i_1,i_2,i_3,l} + \frac{\lambda}{\left(\Delta x_1\right)^2} T_{i_1+1,i_2,i_3,l} + \\
&+ \frac{\lambda}{\left(\Delta x_2\right)^2} T_{i_1,i_2-1,i_3,l} - \frac{2\lambda}{\left(\Delta x_2\right)^2} T_{i_1,i_2,i_3,l} + \frac{\lambda}{\left(\Delta x_2\right)^2} T_{i_1,i_2+1,i_3,l} + \\
&+ \frac{\lambda}{\left(\Delta x_3\right)^2} T_{i_1,i_2,i_3-1,l} - \frac{2\lambda}{\left(\Delta x_3\right)^2} T_{i_1,i_2,i_3,l} + \frac{\lambda}{\left(\Delta x_3\right)^2} T_{i_1,i_2,i_3+1,l} = \\
&= \frac{\rho c}{\Delta t} T_{i_1,i_2,i_3,l} - \frac{\rho c}{\Delta t} T_{i_1,i_2,i_3,l-1}
\end{aligned} \tag{5}$$

in each time step $l$ [2].
Then, for example $1 \le i_1 \le 3,\ 1 \le i_2 \le 3,\ 1 \le i_3 \le 3$ we obtain

$$\begin{aligned}
\det A_3 = \det A_1 \cdot \det\left(A_1^2 - 2d_1^2 I_1\right) \cdot \det\left(A_1^2 - 2d_2^2 I_1\right) \cdot \\
\cdot \det\left(A_1^2 - 2\left(d_2 + d_1\right)^2 I_1\right) \cdot \det\left(A_1^2 - 2\left(d_2 - d_1\right)^2 I_1\right)
\end{aligned} \tag{6}$$

Using the characteristic polynomials of the matrix $A_1^2$ we get the following form

$$\begin{aligned}
&\det A_3 = \\
&= \det A_1 \cdot W_{A_1^2}\left(2d_1^2\right) \cdot W_{A_1^2}\left(2d_2^2\right) \cdot W_{A_1^2}\left(2\left(d_2 + d_1\right)^2\right) \cdot W_{A_1^2}\left(2\left(d_2 - d_1\right)^2\right)
\end{aligned} \tag{7}$$

If $f$ is any monic polynomial of one variable, $A$ is any square matrix of degree $n$, $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of the matrix $A$ the matrix $f(A)$ has eigenvalues $f(\lambda_1), f(\lambda_2), ..., f(\lambda_n)$ [3].

So, the characteristic polynomial of matrix $f(A)$ has the following form

$$\begin{aligned} W_{f(A)}(\lambda) &= (-1)^n \left[\lambda - f(\lambda_1)\right] \cdot \left[\lambda - f(\lambda_2)\right] \cdot ... \cdot \left[\lambda - f(\lambda_n)\right] = \\ &= (-1)^n \Big[\lambda^n - \tau_1(\lambda_1, \lambda_2, ..., \lambda_n)\lambda^{n-1} + \tau_2(\lambda_1, \lambda_2, ..., \lambda_n)\lambda^{n-2} + ... + \\ &\quad + (-1)^n \tau_n(\lambda_1, \lambda_2, ..., \lambda_n)\Big] \end{aligned} \tag{8}$$

where $\tau_1(\lambda_1, \lambda_2, ..., \lambda_n)$, $\tau_2(\lambda_1, \lambda_2, ..., \lambda_n)$,..., $\tau_n(\lambda_1, \lambda_2, ..., \lambda_n)$ are the fundamental symmetric polynomials [4].

Using the fact above we determine the particular characteristic polynomials

$$W_{A_1^2}\left(2d_1^2\right), W_{A_1^2}\left(2d_2^2\right), W_{A_1^2}\left(2\left(d_2 + d_1\right)^2\right), W_{A_1^2}\left(2\left(d_2 - d_1\right)^2\right) \tag{9}$$

In our case $n = 3$ and

$$\begin{aligned} \tau_1(\lambda_1, \lambda_2, \lambda_3) &= \lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr} A_1 = 3a_1 \\ \tau_2(\lambda_1, \lambda_2, \lambda_3) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 3a_1^2 - 2b_1^2 \\ \tau_3(\lambda_1, \lambda_2, \lambda_3) &= \lambda_1\lambda_2\lambda_3 = \det A_1 = a_1^3 - 2a_1 b_1^2 \end{aligned} \tag{10}$$

Then

$$\begin{aligned} &W_{A_1^2}(p) = \\ &= -\left[p^3 - \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)p^2 + \left(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2\right)p - \lambda_1^2\lambda_2^2\lambda_3^2\right] \end{aligned} \tag{11}$$

Designating the following order

$$\begin{aligned} &\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \left(\lambda_1 + \lambda_2 + \lambda_3\right)^2 - 2\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) = 3a_1^2 + 4b_1^2 \\ &\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 = \left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)^2 - 2\lambda_1\lambda_2\lambda_3\left(\lambda_1 + \lambda_2 + \lambda_3\right) = \\ &\qquad = 3a_1^4 + 4b_1^4 \\ &\lambda_1^2\lambda_2^2\lambda_3^2 = \left(\lambda_1\lambda_2\lambda_3\right)^2 = \left(a_1^3 - 2a_1b_1^2\right)^2 \end{aligned} \tag{12}$$

and putting in place $p$ adequate values from (9) and (12) we obtain

$$
\begin{aligned}
&W_{A_1^2}\left(2d_1^2\right) = -8d_1^6 + 4\left(3a_1^2 + 4b_1^2\right)d_1^4 - 2\left(3a_1^4 + 4b_1^4\right)d_1^2 + \left(a_1^3 - 2a_1b_1^2\right)^2 \\
&W_{A_1^2}\left(2d_2^2\right) = -8d_2^6 + 4\left(3a_1^2 + 4b_1^2\right)d_2^4 - 2\left(3a_1^4 + 4b_1^4\right)d_2^2 + \left(a_1^3 - 2a_1b_1^2\right)^2 \\
&W_{A_1^2}\left[2\left(d_2 + d_1\right)^2\right] = -8\left(d_2 + d_1\right)^6 + 4\left(3a_1^2 + 4b_1^2\right)\left(d_2 + d_1\right)^4 + \\
&\qquad -2\left(3a_1^4 + 4b_1^4\right)\left(d_2 + d_1\right)^2 + \left(a_1^3 - 2a_1b_1^2\right)^2 \\
&W_{A_1^2}\left[2\left(d_2 - d_1\right)^2\right] = -8\left(d_2 - d_1\right)^6 + 4\left(3a_1^2 + 4b_1^2\right)\left(d_2 - d_1\right)^4 + \\
&\qquad -2\left(3a_1^4 + 4b_1^4\right)\left(d_2 - d_1\right)^2 + \left(a_1^3 - 2a_1b_1^2\right)^2
\end{aligned}
\tag{13}
$$

So, the determinant (7) is in the form

$$
\begin{aligned}
&\det A_3 = (a_1^3 - 2a_1b_1^2)\cdot \\
&\cdot[(2d_1^2)^3 - (3a_1^2 + 4b_1^2)(2d_1^2)^2 + 2d_1^2(3a_1^4 + 4b_1^4) - (a_1^3 - 2a_1b_1^2)^2]\cdot \\
&\cdot[(2d_2^2)^3 - (3a_1^2 + 4b_1^2)(2d_2^2)^2 + 2d_2^2(3a_1^4 + 4b_1^4) - (a_1^3 - 2a_1b_1^2)^2]\cdot \\
&\cdot\{[2(d_1 + d_2)^2]^3 - (3a_1^2 + 4b_1^2)[2(d_1 + d_2)^2]^2 + \\
&+2(3a_1^4 + 4b_1^4)(d_1 + d_2)^2 - (a_1^3 - 2a_1b_1^2)^2\}\cdot \\
&\cdot\{[2(d_1 - d_2)^2]^3 - (3a_1^2 + 4b_1^2)[2(d_1 - d_2)^2]^2 + \\
&+2(3a_1^4 + 4b_1^4)(d_1 - d_2)^2 - (a_1^3 - 2a_1b_1^2)^2\}
\end{aligned}
\tag{14}
$$

Finally, after transformation we have the following form of this determinant

$$
\begin{aligned}
\det A_3 = &[a_1^2 - 2(d_1 + d_2)^2]a_1[a_1^2 - 2(b_1 + d_1 - d_2)^2][a_1^2 - 2(b_1 + d_2)^2]\cdot \\
&\cdot[a_1^2 - 2(b_1 - d_2)^2][a_1^2 - 2(b_1 + d_1 + d_2)^2](-2d_2^2 + a_1^2)\cdot \\
&\cdot[a_1^2 - 2(b_1 + d_1)^2][a_1^2 - 2(b_1 - d_1 - d_2)^2][a_1^2 - 2(b_1 - d_1)^2]\cdot \\
&\cdot(-2b_1^2 + a_1^2)[a_1^2 - 2(b_1 - d_1 + d_2)^2][a_1^2 - 2(d_1 - d_2)^2](a_1^2 - 2d_1^2)
\end{aligned}
\tag{15}
$$

## Conclusion

The procedure given in this article constitutes a special case ($n = 3$) of the general procedure for calculating the determinants of the block band matrix appearing in the $n$-dimensional Fourier equation when the FDM is used.

## References

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