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PROPERTIES OF A SET OF IDEMPOTENT ELEMENTS OF GENERALISED INVERSE SEMIGROUPS

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Abstract. In the last paper we considered properties of partial order in a generalised inverse semigroup. In this paper we show that a set of idempotent elements of a generalised inverse semigroup is a commutative generalised inverse semigroup. The next problem which appears is if the only commutative inverse semigroups are these which consist of idempotent elements. We prove that it is not the case. We show the commutative inverse semigroup which do not consist only of idempotent elements. We also demonstrate that this inverse generalised semigroup is isomorphic to Ehresmann's pseudogroup.

Introduction

The notion of a pseudogroup was formed progressively together with the development of differential geometry. The first mathematicians who realized that the classic notion of a group of transformations was not sufficient for purposes of differential geometry were O. Veblen and J.H.C. Whitehead in 1932. Their definition was improved by J.A. Schouten and J. Haantjes in 1937, S. Gołąb in 1939 and C. Ehresmann in 1947. Gołąb's and Ehresmann's definitions are still applicable. It was shown in [1] that axioms of Ehresmann's definition can be formulated in the equivalent way which implies proofs. We used this definition in [2] to show that a group of transformations can be treated as a pseudogroup.

Main result

Let us recall the following version of Ehresmann's definition which is used in differential geometry and can be found in [3].

Definition 1. *A pseudogroup of transformations on a topological space S is a set Γ of transformations satisfying the following axioms:*

1° *Each $f \in \Gamma$ is a homeomorphism of an open set of S onto another open set of S;*

2° *If $f \in \Gamma$, then the restriction of f to an arbitrary open subset of the domain of f is in Γ;*

3° *Let* $U = \bigcup_i U_i$ *where each* U_i *is an open set of* S. *A homeomorphism* f *of* U *onto an open set of* S *belongs to* Γ, *if the restriction of* f *to* U_i *is in* Γ *for every* i;

4° *For every open set* U *of* S, *the identity transformation of* U *is in* Γ;

5° *If* $f \in \Gamma$, *then* $f^{-1} \in \Gamma$*;*

6° *If* $f \in \Gamma$ *is a homeomorphism of* U *onto* V *and* $g \in \Gamma$ *is a homeomorphism of* Y *onto* Z *and if* $V \cap Y$ *is non-empty, then the homeomorphism* $g \circ f$ *of* $f^{-1}(V \cap Y)$ *onto* $g(V \cap Y)$ *is in* Γ.

We will also use the following definition introduced in [1].

Definition 2. *A non-empty set* Γ *of functions, for which domains* D_f *are arbitrary non-empty sets, can be called a pseudogrup of functions if it satisfies the following conditions*:

1° $f(D_f) \cap D_g \neq \varnothing \Rightarrow g \circ f \in \Gamma$ for $f, g \in \Gamma$,

2° $f^{-1} \in \Gamma$ for $f \in \Gamma$,

3° $\bigcup \Gamma' \in \Gamma$ for $\Gamma' \in \langle \Gamma \rangle$

where

$$\langle \Gamma \rangle = \{\varnothing \neq \Gamma' \subset \Gamma \ : \cup \Gamma' \textit{ is a function and } \cup (\Gamma')^{-1} \text{ is a } \textit{function}\}$$

and

$$(\Gamma')^{-1} = \{f^{-1} \ : \ f \in \Gamma'\}$$

and f^{-1} denotes an inverse relation.

It was shown in [1] that if Γ is a pseudogrup, then $(\bigcup_{f \in \Gamma} D_f, \{D_f : f \in \Gamma\} \cup \{\varnothing\})$ is a topological space and Γ is an Ehresmann pseudogrup of transformations on this topological space. On the other hand, if Γ is an Ehresmann pseudogrup of transformations on a topological space S, then Γ is a pseudogrup of functions.

We will use the following definition which we can find in [4].

Definition 3. *We will say that* Γ *is a Schouten-Haantjes pseudogroup if it satisfy the following axioms:*

1° *If* $f \in \Gamma$, $g \in \Gamma$ and $g \circ f$ *is defined then* $g \circ f \in \Gamma$,

2° *If* $f \in \Gamma$ and f^{-1} *is defined then* $f^{-1} \in \Gamma$.

We will need the following definition which was introduced in [5].

Definition 4. *A generalized inverse semigroup is a partial groupoid* $(B, \bullet)$ *satisfying the following axioms:*

1° $a \bullet (b \bullet c) = (a \bullet b) \bullet c$

holds when one of the sides is defined;

2° *For every* $a \in B$ *there exists exactly one* $b \in B$ *such that*

$$a \bullet (b \bullet a) = a \quad \textit{and} \quad b \bullet (a \bullet b) = b$$

We will also need for elements of a generalized inverse semigroup $(B, \bullet)$ the following definitions and denotations which were introduced in [5]. We will write ab instead of $a \bullet b$. For every $a \in B$ the only one $b \in B$ from 2° of Definition 4 will be denoted by a' and called a generalised inverse element of a, $a'a$ will be called a right identity of a and aa' a left identity of a. It is obvious that a will be then a generalised inverse element of a', $a'a$ will be a left identity of a' and aa' a right identity of a'. If a is a right and left identity for all elements of B we say that a is an identity. We will say that $a \in B$ is an idempotent element when $aa = a$. It was shown in [5] that $a = a'$ for an idempotent element a, so it means that its generalised inverse element, right and left identity are all equal to a. It was also shown that the following relation:

$$a \le b \Leftrightarrow ba'a = a \tag{1}$$

is a partial order in a generalised inverse semigroup. To prove it we used a lemma saying that if a,b are idempotent elements the operation ab is commutative.

It was proved in [6] that we can obtain an inverse semigroup from every generalised inverse semigroup $(B, \bullet)$ joining an element $O \notin B$. Then $(B \cup \{O\}, *)$ is a semigroup where the operation $*$ is defined in the following way:

$a * b = ab$ when the operation ab is defined;

O in the other case.

We will also use the theorem which was proved in [7] and says that if Γ is a pseudogroup of transformations on a topological space S then Γ is a generalised inverse semigroup with identity. Obviously we can replace a pseudogroup of transformations by a pseudogroup of functions and the theorem will be true. It was shown in [7] that even a Schouten-Haantjes pseudogroup is a generalized inverse semigroup. As the definition of Schouten-Haantjes is more general, we can also say that a pseudogroup of functions is a generalized inverse semigroup. It was also proved in [8] that every generalised inverse semigroup is isomorphic to a Schouten- -Haantjes pseudogroup.

Now we will formulate the problems. Which generalised inverse semigroups are commutative? We can formulate the following theorem:

Theorem 1. *The set of all idempotent elements of a generalised inverse semigroup is a commutative generalised inverse semigroup.*

Proof. As idempotent elements belong to a generalised inverse semigroup, $1°$ of Definition 4 will be satisfied. As we noticed above, idempotent elements are equal in their left identities, right identities and inverse elements. It means that $2°$ of Definition 4 is satisfied. We also noticed above that the operation in the set of idempotent elements is commutative.

The next problem appears if the commutative inverse semigroups are only these which consist of idempotent elements. We grove that it is not the case.

Example 1. *Let us consider the set which consists of two functions: identity defined on* $(0,\div\infty)$ *and* $\frac{1}{x}$ *defined also on* $(0,\div\infty)$. *It is a commutative inverse semigroup but* $\frac{1}{x}$ *is not an idempotent element.*

Proof. It is not only a Schouten-Haantjes pseudogroup, but also an Ehresmann's pseudogroup. It is easier to check it using Definition 2. Obviously the topology will be antidiscrete. As it was shown in [7] that even a Schouten-Haantjes pseudogroup is a generalized inverse semigroup we have a generalised inverse semigroup, but we notice that the operation is defined for all two elements. So it is also an inverse semigroup. That is required.

Conclusions

In [9] we considered properties of partial order in generalised inverse semigroups. Now we considered properties of a set of idempotent elements. We proved that if a generalised inverse semigroup consists of idempotent elements it is commutative. We also show that the inverse theorem is not true. There exist generalised inverse semigroups which are commutative although they do not consist only of idempotent elements.

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