ON RIGHT HEREDITARY SPSD-RINGS OF BOUNDED REPRESENTATION TYPE I

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Abstract. The structure of right hereditary semiperfect semidistributive rings of bounded representation type is described in terms of Dynkin diagrams and diagrams with weights. We describe it using a reduction to mixed matrix problems.

Introduction

This paper is devoted to the study of boundedness of right hereditary semiperfect semidistributive rings (SPSD-rings, in short) considered in [1] and is a continuation of it. These rings were first described in [2].

We use notation and definitions of articles [1, 3, 4] and books [5, 6].

Recall that ring $A$ has a bounded representation type if there is an upper bound on the number of generators required for indecomposable finitely presented $A$-modules. Otherwise it is of the unbounded representation type.

In this paper we prove the necessity of the following main theorem which gives the structure of right hereditary SPSD-rings of bounded representation type in terms of Dynkin diagrams and diagrams with weights:

**Theorem 1.** Let $\{O_i\}$ be a family of discrete valuation rings with a common skew field of fractions $D$, and let $S = S_0 \cup S_1$ be a disjoint union of subposets. A right hereditary SPSD-ring $A$ is of bounded representation type if and only if $A = A(S,O)$ and the undirected graph $\mathcal{D}(S)$ of the Hasse diagram $\mathcal{H}(S)$ of the poset $S$ is a finite disjoint union of Dynkin diagrams of the type $A_n, D_n, E_6, E_7, E_8$ and the following diagrams with weights:

1. $\begin{align*}
\circ & \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \\
\circ & \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots
\end{align*}$

2. $\begin{align*}
\circ & \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \\
\circ & \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots
\end{align*}$

3. $\begin{align*}
\bullet & \leftarrow \circ \rightarrow \bullet \rightarrow \bullet
\end{align*}$
where all vertices with weight 1 correspond to the minimal elements of the poset S.

Note that this theorem was first formulated in [7], where it is given without proof, and it can be considered as a simple generalization of [8, Theorem 1]. In this paper we give two different proofs of the necessity of this theorem using the results of [3, 8, 9].

All rings considered in this paper are assumed to be associative (but not necessary commutative) with \(1 \neq 0\), and all modules are assumed to be unital.

1. Preliminaries

According Gabriel [10] and Dlab and Ringel [11], a hereditary finite dimensional algebra is of finite representation type if and only if the corresponding diagram is a Dynkin diagram of the type \(A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4\) or \(G_2\). From this fundamental result [1, Theorem 3] and [1, Proposition 4] we immediately obtain the following statement:

**Proposition 2.** If diagram \(\Gamma(S)\) of a poset \(S\) is not a disjoint union of the Dynkin diagrams of the type \(A_n, D_n, E_6, E_7, E_8\), then the right hereditary SPSD-ring \(A = A(S, O)\) is a ring of the unbounded representation type.

**Proposition 3.** If ring \(A\) is of the bounded representation type then each of its minors is of bounded representation type too.

**Proof.** Let \(P\) be a finitely generated projective \(A\)-module, \(B = \text{End}_A(P)\), and let \(M\) be a finitely presented \(B\)-module. Then there are the exact sequences:

\[
0 \rightarrow X \rightarrow B^m \rightarrow M \rightarrow 0 \tag{1}
\]

\[
0 \rightarrow Y \rightarrow B^n \rightarrow X \rightarrow 0 \tag{2}
\]

Denote by \(C(P)\) the full subcategory of the category of all \(A\)-modules consisting of \(A\)-modules \(M\) such that there exists an exact sequence

\[
P^j \rightarrow P^i \rightarrow M \rightarrow 0 \tag{3}
\]

where \(P^i\) denotes a direct sum of modules isomorphic to \(P\). By the Morita theorem [5, Theorem 10.7.2], it follows that the categories \(B\)-mod and \(C(P)\) are equivalent,
hence there is an $A$-module $M' \in C(P)$ such that $M = F(M') = \text{Hom}_A(P, M')$. Therefore there is a sequence

$$0 \to X \to B^n \to F(M') \to 0$$  \hspace{1cm} (4)

Applying the exact functor $G = \ast \otimes_B P$ to the exact sequences (2) and (4), we get

$$0 \to G(X) \to P^n \to M' \to 0$$
$$0 \to G(Y) \to P^m \to G(X) \to 0$$  \hspace{1cm} (5)

Hence $\mu_1(M) = \mu_1(P^n) - \mu_1(G(X)) = ns - \mu_1(G(X))$, where $s = \mu_1(P)$, and $\mu_1(U)$ is the minimum number of generators of an $A$-module $U$.

Since $A$ is a ring of the bounded representation type, there exists a number $N$ such that $\mu_1(U) \leq N$ for any $A$-module $U$. Therefore $\mu_1(M) \leq N$ and $\mu_1(G(X)) \leq N$, that is, $ns = \mu_1(M) + \mu_1(G(X)) \leq 2N$, i.e. $n \leq 2N/s$. Writing $2N/s = N_1$ we obtain that $\mu_0(M) \leq n \leq N_1$ and this is true for any finitely presented $B$-module $M$. Therefore $B$ is a ring of bounded representation type.

2. Mixed matrix problems and posets

Let $O$ be a discrete valuation ring with a skew field of fractions $D$ and the Jacobson radical $R = \pi O = O\pi$.

By left $O$-elementary transformations of rows of a matrix $X$ we mean the transformations of the following three types:

(a) interchanging of two rows;
(b) multiplications of a row on the left by an invertible element of $O$;
(c) addition of a row multiplied on the left by an arbitrary element of $O$ to another row.

In a similar way we define left $D$-elementary transformations of rows and, by symmetry, right $O$-elementary and right $D$-elementary transformations of columns.

Let $T = (T_{ij})$ be a rectangular matrix with entries in $D$ partitioned into $n$ horizontal strips $T_{ij}; i = 1, \ldots, n$ and $m$ vertical strips $T_{ij}; j = 1, \ldots, m$ so that a block $T_{ij}$ is the intersection of the $i$-th horizontal strip $T_i$ and the $j$-th vertical strip $T^j$.

Let $M_n(D)$ be the ring of $n \times n$ matrices over $D$ with matrix units $e_{ij}$. Following [8] we consider a transforming algebra $X = \bigoplus_{i,j=1}^n e_{ij}X_{ij} \subseteq M_n(D)$ such that:

(a) $X_{ii} = O$ or $X_{ii} = D$;
(b) $X_{ij}X_{jk} \subseteq X_{ik}$
(c) $X_{ij}X_{ij} \neq X_{ij}$ for $i \neq j$.

for each $i,j,k = 1, 2, \ldots, n$.

Obviously, $X_{ij} = D$ or $X_{ij} = \pi^{\alpha_{ij}}O$, where $\alpha_{ij} \in \mathbb{Z}$. We set $D = \pi^{\alpha}O$ and $0 = \pi^{\alpha}O$. 


A family of elementary transformations over the row strips of a rectangular matrix $T = (T_{ij})$ of the following form:
(i) left $X_{i'}$-elementary transformations of rows in the strip $T_i$;
(ii) addition of rows in a strip $T_i$ multiplied on the left by elements of $X_{i j}$ to rows of a strip $T_j$.

These will be called **admissible transformations** with respect to an algebra $X$.

In a similar way one can define admissible transformations over the column strips of a matrix $T$ with respect to an algebra $Y = \bigoplus_{i,j=1}^{m} e_{ij} Y_{ij} \subseteq M_m(D)$.

The **dimension** of a stripped matrix $T$ is the vector
$$d = d(T) = (d_1, d_2, \ldots, d_n, d^1, d^2, \ldots, d^m),$$
where $d_i$ is the number of rows of the $i$-th horizontal strip $T_i$, and $d^j$ is the number of columns of the $j$-th vertical strip $T^j$ for $j = 1, \ldots, m$. We set
$$\dim(T) = \sum_{i=1}^{n} d_i + \sum_{j=1}^{m} d^j.$$  

According to [9], a mixed matrix problem has a **bounded representation type**, if there is a constant $C$ such that $\dim(T) < C$ for all indecomposable matrices $T$.

**Flat mixed matrix problem:**
Given a triangular matrix $T = (T_{ij})$ with entries in $D$ partitioned into $n$ horizontal strips $\{T_i\}_{i=1,\ldots,n}$ and $m$ vertical strips $\{T^j\}_{j=1,\ldots,m}$, two transforming algebras $X \subseteq M_n(D)$ and $Y \subseteq M_m(D)$. One performs admissible transformations over row strips with respect to $X$ and admissible transformations over column strips with respect to $Y$. Define the boundedness type of this matrix problem.

This matrix problem was solved in [9] in terms of critical pairs of sets in the sense of Kleiner [12].

Recall that a totally ordered set consisted of $n$ elements is called a **chain** and denoted by $(n)$. A cardinal sum of $k$ chains consisting of $n_1, n_2, \ldots, n_k$ elements is denoted by $(n_1, n_2, \ldots, n_k)$. A cardinal sum of posets $P$ and $Q$ is denoted by $P \leq Q$. Denote by $N$ the poset $(a < b > c < d)$.

Associate with a transforming algebra $X$ a poset $P(X) = \sum_{i=1}^{n} P_i$, which is a cardinal sum of posets $P_i$, where $P_i$ is a chain of the following type:
(a) $P_i = \{ p_i^0 \}$ is a one-point chain if $X_{ii} = D$;
(b) $P_i = \{ p_i^k \}_{k \in \mathbb{Z}}$ is an infinite chain if $X_{ii} = O$.

The order relation in $P(X)$ is defined as follows:
$$p_i^k \leq p_i^l \iff k - l \geq \alpha_y \text{ if } X_{ij} = \alpha_y^\alpha O.$$  

(8)
Definition 1. A pair \((P,Q)\) of posets is called a **critical pair of sets** (in the sense of Kleiner) if one of the following conditions is satisfied up to the transposition of \(P\) and \(Q\):

- \(P = (1)\); \(Q = (1,1,1) \lor (2,2,2) \lor (1,3,3) \lor (1,2,5) \lor N \leq 4\);
- \(P = (2)\); \(Q = (1,1,1) \lor (3,3) \lor (2,5)\);
- \(P = (3)\); \(Q = (2,2) \lor (1,5)\);
- \(P = (4)\); \(Q = (1,3)\);
- \(P = (5)\); \(Q = N\);
- \(P = (6)\); \(Q = (1,2)\);
- \(P = (1,1)\); \(Q = (1,1)\).

**Theorem 4** [9]. A flat matrix problem defined by a pair of transforming algebras \((X,Y)\) of the above type is of bounded representation type if and only if the pair of partially ordered sets \((P(X), P(Y))\) contains no critical pairs of sets in the sense of Kleiner.

3. Proof of the necessity in Theorem 1

**Lemma 5.** Let \(O\) be a discrete valuation ring with a skew field of fractions \(D\) and the radical \(R = \pi O = O\pi\). Then the ring

\[
A = \begin{pmatrix}
O & D & D & D \\
0 & D & 0 & 0 \\
0 & 0 & D & 0 \\
0 & 0 & 0 & D
\end{pmatrix},
\]

(10)

is a ring of unbounded representation type.

**Proof.** Let \(M\) be a finitely generated right \(A\)-module that is given by the set \(\{t; l_1, l_2, l_3; T\}\), in which the matrix \(T\) has the following form:

\[
\begin{array}{cccc}
E & T_{12} & T_{13} & T_{14} \\
O & E & O & O \\
O & O & E & O \\
O & O & O & E
\end{array}
\]
where $T_{i} \in M_{r \times l}(D) \ (i = 2,3,4)$ are matrices over $D$. The matrix of transformations $U$ has the following form:

\[
\begin{array}{cccc}
U_{11} & O & O & O \\
O & U_{22} & O & O \\
O & O & U_{33} & O \\
O & O & O & U_{44}
\end{array}
\]

where $U_{11}$ is an invertible matrix with entries in $O$, and $U_{ii} \ (i = 2,3,4)$ are invertible matrices with entries in $D$. Reducing the matrix $T$ by the matrix $U$ leads to the following matrix problem, given by a matrix $T_{1}$

\[
\begin{pmatrix}
A_1 & A_2 & A_3
\end{pmatrix}
\]

and the following admissible transformations:

(a) left $O$-elementary transformations of rows of the matrix $T_{1}$;

(b) right $D$-elementary transformations of columns inside any vertical strip $A_{i} \ (i = 1,2,3)$.

Set

\[
A_{1} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_{2} = \begin{pmatrix}
\pi^{-2} & 0 & \cdots & 0 \\
0 & \pi^{-4} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \pi^{-2n}
\end{pmatrix}, \quad A_{3} = \begin{pmatrix}
\pi^{n-1} \\
\pi^{n-2} \\
\vdots \\
1
\end{pmatrix}
\]

where $\pi \in R = \text{rad } O$, $\pi \neq 0$. By [4, Lemma 3], the matrix $T_{1}$ is indecomposable and therefore ring $A$ is of unbounded representation type.

**Remark 1.**

A mixed matrix problem over a matrix $T_{1}$ is defined by two transforming algebras:

\[
X = O, \quad Y = \begin{pmatrix}
D & 0 & 0 \\
0 & D & 0 \\
0 & 0 & D
\end{pmatrix}
\]
Correspondingly, \( P(X) \) is an infinite chain, and \( P(Y) = \{1, 1, 1\} \). Therefore the pair of posets \( \{P(X), P(Y)\} \) contains a critical pair of sets \( \{(2), (1, 1, 1)\} \). By theorem 4 this matrix problem is of unbounded representation type.

**Lemma 6.** Let \( O \) be a discrete valuation ring with a skew field of fractions \( D \) and the radical \( R = \pi O = O\pi \). Then the ring

\[
A = \begin{pmatrix}
O & 0 & 0 & D \\
0 & O & 0 & D \\
0 & 0 & D & D \\
0 & 0 & 0 & D \\
\end{pmatrix},
\]

(13)

corresponding to the diagram

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \quad \bullet
\end{array}
\]

is a ring of the unbounded representation type.

**Proof.** Let \( M \) be a finitely generated right \( A \)-module which is given by the set \( \{t_1, t_2; l_1, l_2; T\} \), in which the matrix \( T \) has the following form:

\[
\begin{array}{cccc}
E & O & O & T_{14} \\
O & E & O & T_{24} \\
O & O & E & T_{34} \\
O & O & O & E
\end{array}
\]

where \( T_{ij} \in M_{l_1 \times l_2}(D) \) \((i = 1, 2)\) and \( T_{34} \in M_{l_1 \times l_2}(D) \) are matrices over \( D \). The matrix of transformations \( U \) has the following form:

\[
\begin{array}{cccc}
U_{11} & O & O & O \\
O & U_{22} & O & O \\
O & O & U_{33} & O \\
O & O & O & U_{44}
\end{array}
\]

where \( U_i \) is an invertible matrix with entries in \( O \) \((i = 1, 2, 3, 4)\) are invertible matrices with entries in \( D \). Reducing the matrix \( T \) by the matrix \( U \) is equivalent to the matrix problem given by a matrix \( T_1 \)
and the following admissible transformations:
(a) left $D$-elementary transformations of rows of the matrix $T_1$;
(b) right $O$-elementary transformations of columns inside any block $A_i$ ($i = 1, 2$);
(c) right $D$-elementary transformations of columns inside the block $A_3$.

Set

\[
A_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad
A_2 = \begin{pmatrix} \pi^2 & 0 & \cdots & 0 \\ 0 & \pi^4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi^{2n} \end{pmatrix}, \quad
A_3 = \begin{pmatrix} 1 \\ \pi \\ \vdots \\ \pi^{n-1} \end{pmatrix}.
\]

(14)

By [4, Lemma 3], the matrix $T_1$ is indecomposable. So the corresponding module $M$ is indecomposable and the ring $A$ is of the unbounded representation type.

Remark 2.
A mixed matrix problem over a matrix $T_1$ is defined by transforming algebras:

\[
X = D, \quad Y = \begin{pmatrix} O & 0 & 0 \\ 0 & O & 0 \\ 0 & 0 & D \end{pmatrix}.
\]

(15)

Correspondingly, we have two posets: $P(X) = (1)$ is a one-point chain (1), and $P(Y)$ is a cardinal sum of two infinite chains and a one-point chain (1). Therefore the pair of posets $\{P(X), P(Y)\}$ contains a critical pair of sets $\{(1), (3,3,1)\}$. By theorem 4 this matrix problem is of the unbounded representation type.

Lemma 7. Let $O$ be a discrete valuation ring with a skew field of fractions $D$ and the radical $R = \pi O = O\pi$. Then the ring

\[
A = \begin{pmatrix} O & 0 & D & D \\ 0 & O & D & D \\ 0 & 0 & D & D \\ 0 & 0 & 0 & D \end{pmatrix},
\]

(16)

corresponding to the diagram

\[
\begin{array}{ccc}
\text{○} & \rightarrow & \text{○} \\
\end{array}
\]

is a ring of the unbounded representation type.
The proof of this lemma is the same as for Lemma 6.

**Lemma 8.** Let $O$ be a discrete valuation ring with a skew field of fractions $D$ and the radical $R = \pi O = O\pi$. Then the ring

$$A = \begin{pmatrix} O & 0 & D & D \\ 0 & D & 0 & D \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix},$$

(17)

corresponding to the diagram

$$\bullet \longrightarrow \odot \longrightarrow \bullet$$
is a ring of the unbounded representation type.

**Proof.** Let $M$ be a finitely generated right $A$-module that is given by the set \{t, l_1, l_2, l_3, T\}, in which the matrix $T$ has the following form:

$$
\begin{array}{cccc}
E & O & T_{13} & T_{14} \\
O & E & O & T_{24} \\
O & O & E & O \\
O & O & O & E \\
\end{array}
$$

where $T_{13} \in M_{t \times l_1}(D)$, $T_{14} \in M_{t \times l_2}(D)$, and $T_{24} \in M_{l_3 \times l_2}(D)$. The matrix of transformations $U$ has the following form:

$$
\begin{array}{cccc}
U_{11} & O & O & O \\
O & U_{22} & O & O \\
O & O & U_{33} & O \\
O & O & O & U_{44} \\
\end{array}
$$

where $U_{11}$ is an invertible matrix with entries from $O$, and $U_{ii}$ ($i = 2,3,4$) are invertible matrices with entries from $D$. Reducing the matrix $T$ by the matrix $U$ is equivalent to the matrix problem given by a matrix $T_1$.

$$
\begin{array}{cc}
A_1 & A_2 \\
O & A_3 \\
\end{array}$$
and the following admissible transformations:
(a) left $O$-elementary ($D$-elementary) transformations of rows inside the first
(second) horizontal strip of the matrix $T_1$;
(b) right $D$-elementary transformations of columns inside each vertical strip of the
matrix $T_1$.

Reducing the matrix $A_1$ to the form

$$
\begin{pmatrix}
E & O \\
O & O \\
\end{pmatrix}
$$

we get that the matrix $A_2$ has the form:

$$
\begin{pmatrix}
B_1 & B_2 \\
\end{pmatrix}
$$

We can add any column of $B_2$ multiplied on the right by elements of $D$ to any
column of $B_1$. Thus the matrices $A_1$, $B_1$ and $B_2$ form the matrix problem considered in [4, Problem II]. By [4, Lemma 4] the ring $A$ is of the unbounded representa-
tion type.

**Remark 3.**
A mixed matrix problem which forms matrices $A_1$, $B_1$ and $B_2$ is defined by
transforming algebras:

$$
X = O, \quad Y = \begin{pmatrix} D & D & 0 \\
0 & D & 0 \\
0 & 0 & D \\
\end{pmatrix} \quad (18)
$$

Correspondingly, we have two posets: $P(X)$ is an infinite chain and $P(Y)$ is a cardi-
nal sum $(1, 2)$. Therefore the pair of posets $\{P(X), P(Y)\}$ contains a critical pair of
sets $\{(6), (1, 2)\}$. By Theorem 4 this matrix problem is of unbounded representa-
tion type.

Analogously, one can prove the following lemma:

**Lemma 9.** Let $O$ be a discrete valuation ring with a skew field of fractions $D$
and the radical $R = \pi O = O\pi$. Then the ring $A$ is of the unbounded representa-
tion type.

$$
A = \begin{pmatrix} O & D & D & D \\
0 & D & 0 & 0 \\
0 & 0 & D & D \\
0 & 0 & 0 & D \\
\end{pmatrix}, \quad (19)
$$
corresponding to the diagram

\[
\begin{array}{c}
\bullet \\
\leftarrow \circ \\
\rightarrow \\
\end{array}
\]

is a ring of the unbounded representation type.

**Lemma 10.** Let \( O \) be a discrete valuation ring with a skew field of fractions \( D \) and the Jacobson radical \( R = \pi O = O \pi \). Then the ring

\[
A = \begin{pmatrix}
O & 0 & 0 & 0 & D \\
0 & D & 0 & 0 & D \\
0 & 0 & D & D & D \\
0 & 0 & 0 & D & 0 \\
0 & 0 & 0 & 0 & D \\
\end{pmatrix}
\] (20)

corresponding to the diagram

\[
\begin{array}{c}
\bullet \\
\circ \\
\leftarrow \\
\rightarrow \\
\end{array}
\]

is a ring of the unbounded representation type.

**Proof.** Let \( M \) be a finitely generated right \( A \)-module which is given by the set \( \{t; l_1, l_2, l_3, l_4; T\} \), in which the matrix \( T \) has the following form:

\[
\begin{array}{cccc}
E & O & O & O \\
O & E & O & O \\
O & O & E & T_{34} \\
O & O & O & E \\
O & O & O & E \\
\end{array}
\]

where \( T_{15} \in M_{i \times i_4}(D), T_{i5} \in M_{i \times i_4}(D), (i = 2, 3) \) and \( T_{34} \in M_{i_2 \times i_3}(D) \) are matrices over \( D \). The matrix of transformations \( U \) has the following form:

\[
\begin{array}{cccc}
U_{11} & O & O & O \\
O & U_{22} & O & O \\
O & O & U_{33} & O \\
O & O & O & U_{44} \\
\end{array}
\]
where $U_{11}$ is an invertible matrix with entries in $O$, and $U_{ii}$ ($i = 2, 3, 4, 5$) are invertible matrices with entries in $D$. Reducing the matrix $T$ by the matrix $U$ leads to the matrix problem given by a matrix $T_1$ partitioned into 3 horizontal strips and 2 vertical strips:

\[
\begin{array}{c}
O & O & O & O & U_{55} \\
\end{array}
\]

and the following admissible transformations:
(a) left $O$-elementary transformations with rows of the first horizontal strip of $T_1$;
(b) left $D$-elementary transformations of rows of the second horizontal strip and third horizontal strip of $T_1$;
(c) right $D$-elementary transformations of columns inside each vertical strip of $T_1$.

Then one can reduce the second horizontal strip to the form:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
\end{array}
\]

and the third horizontal strip to the form:

\[
\begin{array}{cccc}
E & 0 & 0 & 0 \\
0 & E & 0 & 0 \\
0 & 0 & E & 0 \\
0 & 0 & 0 & E \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\begin{array}{c}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\begin{array}{c}
0 & 0 & E & 0 \\
0 & 0 & 0 & E \\
0 & 0 & 0 & E \\
0 & 0 & 0 & E \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\begin{array}{c}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

This process leads to the matrix problem given by a matrix $B$ partitioned into 6 vertical strips

\[
\begin{array}{ccccccc}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
\end{array}
\]

and the following admissible transformations:
(a) left $O$-elementary transformations with rows of the first horizontal strip of $B$;
(b) right $D$-elementary transformations with columns inside of each vertical strip $B_i$ ($i = 1,2,...,6$);
(c) right $D$-elementary transformations with columns of vertical strips $B_i$ which are in one-to-one relation with the following poset $S$:

\[
\begin{array}{cccc}
1 & & & 2 \\
& 3 & & \\
4 & & 5 & 6 \\
\end{array}
\]

i.e., if $\alpha_i \leq \alpha_j$ in the poset $S$ then any column of the block $B_i$ can be added to any column of the $B_j$.

It is easy to see the blocks $B_2$, $B_3$, $B_4$ form the matrix problem II, considered in [4]. Therefore ring $A$ is of unbounded representation type.

**Remark 4.**

A mixed matrix problem which forms matrices $B_i$ ($i = 1,...,6$) is defined by two posets: $P(X)$ is an infinite chain, and $P(Y) = S$. Therefore the pair of posets $\{P(X), P(Y)\}$ contains a critical pair of sets $\{(6), (1, 2)\}$. By Theorem 4 this matrix problem is of unbounded representation type.

**Lemma 11.** Ring $A$ corresponding to the diagram

\[\begin{array}{ccc}
\odot & \rightarrow & \hspace{1cm} \\
\rightarrow & & \rightarrow
\end{array}\]

with arbitrary directions of arrows is a ring of unbounded representation type.

**Conclusions**

This paper proves the necessity in Theorem 1 for the ring $A(S, O)$, when all discrete valuation rings corresponding to minimal elements of the poset $S$ are the same. In this case the necessity follows from Lemmas 5-11 and Propositions 2 and 3.

**References**

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