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STURM-LIOUVILLE EIGENVALUE PROBLEMS WITH *MATHEMATICA*

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Abstract. In this paper, the power series method is applied to a class of Sturm-Liouville eigenproblems. The procedure of a symbolic solution to the problem in the *Mathematica* program for Airy's differential equation representing the considered class of S-L equations is presented. An example of numerical results is given.

Introduction

The Sturm-Liouville theory deals with the second-order differential equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+\left(q(x)+\lambda\, r(x)\right)y=0, \quad x\in(a,b) \tag{1}$$

where p, q, r and p' are real and continuous on interval (a,b) known functions and λ is a constant. Equation (1) comprises a wide class of ordinary differential equations of the second order. For example, the equation

$$a_2(x)y''+a_1(x)y'+\left[a_0(x)+\lambda\right]y=0 \tag{2}$$

can be written in the form of equation (1) assuming:

$$p(x)=\exp\left(\int\frac{a_1(x)}{a_2(x)}dx\right),\quad q(x)=\frac{a_0(x)}{a_2(x)}p(x),\quad r(x)=\frac{p(x)}{a_2(x)}$$

Sturm-Liouville (S-L) equations arise by using the method of separation of variables to solve linear partial differential equations (for instance, heat conduction equation, wave equation). Equation (1) is completed by the boundary conditions

$$\alpha_1 y(a)+\alpha_2 y'(a)=0,\ \beta_1 y(b)+\beta_2 y'(b)=0 \tag{3}$$

where α_1, α_2, β_1, β_2 are real numbers.

An example of S-L differential equation is Airy's equation which occurs in the study of the diffraction of light and the deflection of a column that bends under its own weight [1]. A collection of examples of S-L differential equations which are connected with problems in applied mathematics is given in paper [2].

In this paper the solution of an eigenvalue problem of the S-L type with ordinary points by using the *Mathematica* program is presented. The power series method for Airy's equation which represents the considered class of S-L problems was applied.

Eigenvalue problem of S-L type

In this section we present the solution of a class of S-L eigenvalue problems assuming an expansion of the solution on an ordinary point. Using the program *Mathematica,* an exact solution and power series solution to the S-L problem with Airy's differential equation has been obtained.

Exact solution of Airy's eigenvalue problem

We consider Airy differential equation in interval [0,1]:

$$-y''(x) + x\,y(x) = a\lambda\, y(x) \tag{4}$$

The general solution of this equation obtained by using *Mathematica* is [3]

```
In[1]    DSolve[-y″[x]+ x y[x]== a λ y[x],y[x],x]
Out[1]   {{y[x]→AiryAi[x-a λ] C[1]+AiryBi[x-a λ] C[2]}}
```

where AiryAi and AiryBi are Airy's functions [1]. The boundary conditions for function *y* give two homogeneous equations

```
In[2]    eq1=y[0]==0
Out[2]   AiryAi[-a λ] C[1]+AiryBi[-a λ] C[2]==0
In[3]    eq2=y[1]==0
Out[3]   AiryAi[1-a λ] C[1]+AiryBi[1-a λ] C[2]==0
```

For a non-trivial solution to the problem, the determinant of the coefficient matrix of the obtained equation system is set equal to zero. The determinant expressed as a function of parameter λ, can be written in the form

```
f[λ_]=AiryAi[-a λ]*AiryBi[1-a λ]-AiryBi[-a λ]*AiryAi[1-a λ]
```

The graph of this function is presented in Figure 1. Zeros λ_i of the function (eigenvalues) are numerically determined by using the bisection method. In Table 1 the first 10 eigenvalues are shown.

Eigenfunction `y[x,i]`, corresponding to eigenvalue λ_i has the form

```
y[x_,i_]=AiryAi[x-a λi] AiryBi[-a λi]-AiryBi[x-a λi] AiryAi[-a λi]
```

The graphs of the first five eigenfunctions are shown in Figure 2.

```
In[4]    Plot[f[λ],{λ,0,100}]
```

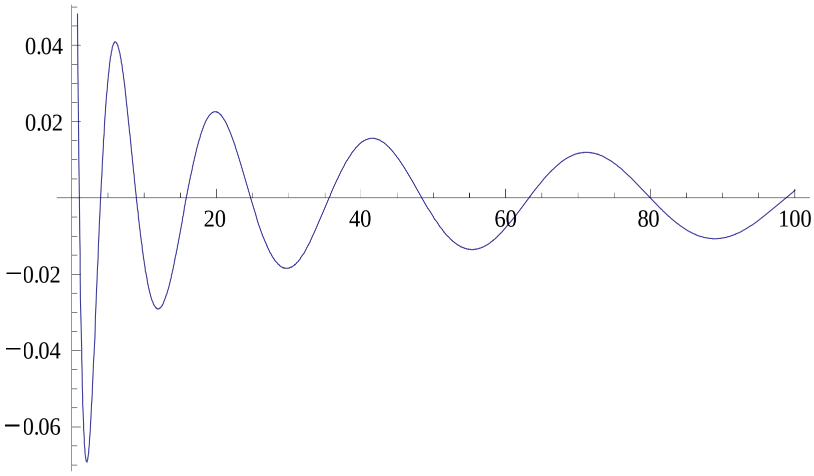

Fig. 1. Graph of function `f[λ]` and 10 roots of characteristic equation

Table 1

Eigenvalues λ_k (k = 1,…, 10) of Airy's problem obtained by exact method and power series method for finite number *n* of terms of the series

k	Power series solution				Exact solution
	n = 10	*n* = 20	*n* = 30	*n* = 40	
1	1.0368	1.03685	1.0368	1.0368	1.0368
2	4.0157	3.99787	3.9979	3.9979	3.9979
3	9.8989	8.93266	8.9327	8.9327	8.9327
4	-	15.8467	15.8414	15.8414	15.8414
5	-	25.0041	24.7240	24.7240	24.7240
6	-	29.6838	35.5819	35.5806	35.5806
7	-	-	48.4752	48.4111	48.4111
8	-	-	58.8992	63.2158	63.2155
9	-	-	65.9097	80.0089	79.9938
10	-	-	-	96.4477	98.7460

Power series solution of Airy's eigenvalue problem

Airy's eigenvalue problem presented in the previous section will be solved now by using the power series method. The solution to the problem with the program *Mathematica* is derived. The numerical calculations for a various number of terms of the series solution were performed. The *Mathematica* procedure of the program for five terms of the series solution is presented below. The procedure can be used in other Sturm-Liouville differential equations with ordinary points.

```
In[5]     Plot[{y[x,1],y[x,2],y[x,3],y[x,4],y[x,5]},{x,0,1}]
```

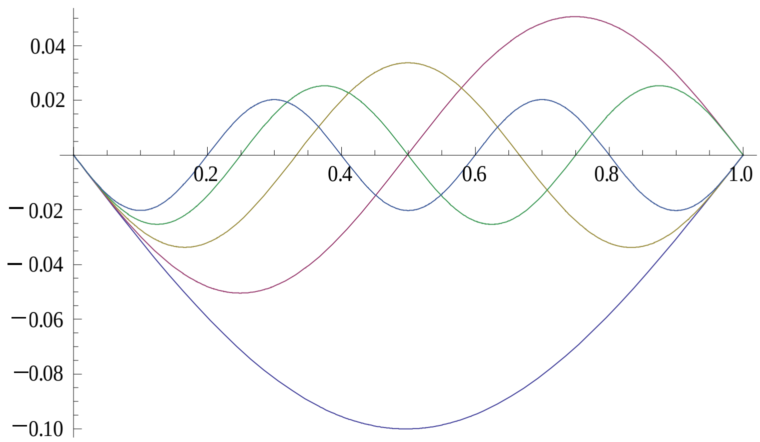

Fig. 2. First five eigenfunctions of considered Airy's eigenproblem

We assume that the solution to the problem has form of the Taylor series

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!}(x - x_0)^k \quad (5)$$

Using the *Mathematica* program, we approximate the solution to the problem by taking into account *n* terms of series (5). In the considered Airy's eigenproblem, we assume $x_0 = \frac{1}{2}$. Therefore, with *Mathematica* for *n* = 5, we obtain:

```
In[6]    sery=Series[y[x],{x,x0,n}]
Out[6]   y[x0]+y'[x0] (x-x0)+1/2 y''[x0] (x-x0)^2+1/6 y^(3)[x0]
  (x-x0)^3+1/24 y^(4)[x0] (x-x0)^4+1/120 y^(5)[x0] (x-x0)^5+O[x-x0]^6
In[7]    sereq=Series[x y[x]-y''[x]-a λ y[x],{x,x0,nn}]
Out[7]   (x0 y[x0]-a λ y[x0]-y''[x0])+(y[x0]+x0 y'[x0]
         -a λ y'[x0]-y^(3)[x0]) (x-x0)+1/2 (2 y'[x0]+x0 y''[x0]
```

```
      -a λ y''[x0]-y(4)[x0]) (x-x0)2+1/6 (3 y''[x0]+x0 y(3)[x0]
      -a λ y(3)[x0]-y(5)[x0]) (x-x0)3+1/24 (4 y(3)[x0]+x0 y(4)[x0]
     -a λ y(4)[x0]-y(6)[x0]) (x-x0)4+1/120 (5 y(4)[x0]+x0 y(5)[x0]
     -a λ y(5)[x0]-y(7)[x0]) (x-x0)5+O[x-x0]6
In[8]    syseqs=LogicalExpand[sereq==0]
Out[8]   x0 y[x0]-a λ y[x0]-y''[x0]==0&&
         y[x0]+x0 y'[x0]-a λ y'[x0]-y(3)[x0]==0&&
         1/2 (2 y'[x0]+x0 y''[x0]-a λ y''[x0]-y(4)[x0])==0&&
         1/6 (3 y''[x0]+x0 y(3)[x0]-a λ y(3)[x0]-y(5)[x0])==0&&
         1/24 (4 y(3)[x0]+x0 y(4)[x0]-a λ y(4)[x0]-y(6)[x0])==0&&
         1/120 (5 y(4)[x0]+x0 y(5)[x0]-a λ y(5)[x0]-y(7)[x0])==0
In[9]    deriv0=Table[D[y[x],{x,i}]/.x→x0,{i,2,n+2}]
Out[9]   {y''[x0],y(3)[x0],y(4)[x0],y(5)[x0],y(6)[x0],y(7)[x0]}
In[10]    sol=Solve[syseqs,deriv0]
Out[10]   {{y(6)[x0]→4 y[x0]+x03 y[x0]-3 a x02 λ y[x0]
         +3 a2 x0 λ2 y[x0]-a3 λ3 y[x0]+6 x0 y'[x0]
        -6 a λ y'[x0],y(7)[x0]→9 x02 y[x0]-18 a x0 λ y[x0]
        +9 a2 λ2 y[x0]+10 y'[x0]+x03 y'[x0]-3 a x02 λ y'[x0]
        +3 a2 x0 λ2 y'[x0]-a3 λ3 y'[x0],y(5)[x0]→4 x0 y[x0]
        -4 a λ y[x0]+x02 y'[x0]-2 a x0 λ y'[x0]
        +a2 λ2 y'[x0],y(4)[x0]→(-x0+a λ) (-x0 y[x0]+a λ y[x0])
        +2 y'[x0],y''[x0]→x0 y[x0]-a λ y[x0],y(3)[x0]→y[x0]
        +x0 y'[x0]-a λ y'[x0]}}
In[11]    sery=Series[y[x],{x,x0,n}]/.sol[[1]]
Out[11]   y[x0]+y'[x0] (x-x0)+1/2 (x0 y[x0]-a λ y[x0]) (x-x0)2
         +1/6 (y[x0]+x0 y'[x0]-a λ y'[x0]) (x-x0)3
         +1/24 ((-x0+a λ) (-x0 y[x0]+a λ y[x0])
        +2 y'[x0]) (x-x0)4+1/120 (4 x0 y[x0]
        -4 a λ y[x0]+x02 y'[x0]-2 a x0 λ y'[x0]
        +a2 λ2 y'[x0]) (x-x0)2 + O[x-x0]6
```

```
In[12]
gesol=Collect[sery,{y[x0],y'[x0],x-
x0}]/.{y[x0]→c1,y'[x0]→c2}
Out[12]  c2 (x+1/12 (x-x0)^4-x0+(x-x0)^3 (x0/6-(a λ)/6)
        +(x-x0)^5 (x0^2/120-(a x0 λ)/60+(a^2 λ2)/120))
        +c1 (1+1/6 (x-x0)^3+(x-x0)^2 (x0/2-(a λ)/2)
      +(x-x0)^5(x0/30-(aλ)/30)+(x-x0)^4(-(1/24) x0 (-x0+aλ)
       +1/24 a λ (-x0+a λ)))
In[13]   a11=Coefficient[gensol,c1]/.x→0
         a12=Coefficient[gensol,c2]/.x→0
         a21=Coefficient[gensol,c1]/.x→1
         a22=Coefficient[gensol,c2]/.x→1
In[14]   x0 = 1/2;  a=10;   det=a11*a22-a21*a12
Out[12]  -(-(95/192)+1/8 (-(1/12)+(aλ)/6)+1/32 (-(1/480)
         +(a λ)/120-(a^2 λ^2)/120)) (49/48+1/4 (1/4-(a λ)/2)
         +1/32 (1/60-(a λ)/30)+1/16 (1/48 (1/2-a λ)
         +1/24 a λ (-(1/2)+a λ)))+(97/192+1/8 (1/12-(a λ)/6)
    +1/32 (1/480-(aλ)/120+(a^2λ^2)/120)) (47/48+1/4 (1/4-(aλ)/2)
         +1/32 (-(1/60)+(a λ)/30)+1/16 (1/48 (1/2-a λ)
         +1/24 a λ (-(1/2)+a λ)))
```

Determinant **det** as a function of parameter λ in a polynomial form is used for the numerical calculations of eigenvalues. The numerical results of the eigenvalues for various numbers n of terms of the series solution and eigenvalues obtained from the exact solution in Table 1 are compared. When the number of terms of the series is too small, then the degree of the polynomial is also small and only a few first eigenvalues can be calculated (see Table 1).

Conclusions

The use of *Mathematica* to solve Sturm-Liouville eigenvalue problems has been presented. Although the eigenvalue equation was derived for a selected eigenproblem (Airy's differential equation), the approach can be used for a class of differential equations with ordinary points. In the case of S-L differential equations with regular singular points, the indicial equation must be first solved and next the procedure presented here can be applied. The numerical calculations of the eigenvalues have shown that the approximate power series solution gives correct results when a sufficiently large number of terms of the series is assumed.

References

[1] Zill D.G., Cullen M.R., Differential Equations with Boundary-Value Problems, Brooks/Cole - Thomson Learning 2005.

[2] Everitt W.N., A Catalogue of Sturm Liouville Differential Equations, [in:] Sturm-Liouville Theory. Past and Present, W.O. Amrein, A.M. Hinz, D.B. Pearson (eds), Birkhäuser-Verlag 2005, 271--331.

[3] Abell M.L., Braselton J.P., Differential Equations with *Mathematica*, Elsevier Academic Press, 2004.