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STATIONARY CHARACTERISTICS OF M^{θ}/M/1 QUEUE WITH SWITCHING OF SERVICE MODES

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Abstract. For an M^{θ}/M/1 queue with a threshold switching of service modes at the start of the service of the next customer an algorithm for determining the stationary distribution of the number of customers and stationary characteristics (average queue length, average waiting time, variance of queue length) is proposed. In the case the minimum number of incoming customers in the group is comparable to threshold value h, the stationary characteristics are found in an explicit form. The results are verified by simulation models constructed with the assistance of GPSS World.

Introduction. Model description

Models of queueing systems where a different intensity of service is used depending on the queue length and where the customers arrive in groups, are frequently used for the study of telecommunication processes [1, 2]. In papers [3, 4] M^{θ}/G/1/m queues with the switching of service modes and threshold blocking of input flow are investigated. Queueing systems with the switching of service modes at the start of the service of the next customer were considered particularly in works [5, 6]. More precisely, in [5] and [6] the stationary distribution of the number of customers for an M/G/1/m queue with dual-speed service and the stationary characteristics of an M/G/1 queue with several switching thresholds are considered, respectively.

In the present paper we study M^{θ}/M/1 and M^{θ}/M/1/m queueing systems with dual-speed service and grouped arrival of customers.

Consider an M^{θ}/M/1 queue without a limitation of queue length. The customers arrive to the system in groups and the time intervals between the successive moments of the arrival of groups of customers are independent random variables with the same exponential distribution with parameter λ. In the n-th group, there is a random number of customers θ_n, which equals k with probability $\mathbf{P}\{\theta_n = k\} = a_k$ $(k \geq 1)$. Assume that the customer groups are served in the order of their arrival and within the groups the customers are served in random order.

Customer service can be performed in two modes. In each of them, the service time is exponentially distributed with parameters μ_1 and μ_2 respectively. While the number of customers in the system does not exceed the given threshold value h ($h \geq 1$), a service is performed in the main mode with intensity μ_1. Switching to a service mode with intensity μ_2 is performed at the time of the beginning of serving the first customer, provided the number of customers exceeds h. As soon as the number of customers becomes less than $h+1$, a reverse switching to a service mode with intensity μ_1 is performed. Assume that $\mu_2 \geq \mu_1$ and denote the system described above by $\mathrm{M}^\theta/\mathrm{M}_1/1$ (index "1" indicates the number of post threshold service modes).

1. Stationary characteristics of $\mathrm{M}^\theta/\mathrm{M}_1/1$ queue

Let $\alpha_1 = \lambda / \mu_1$, $\alpha_2 = \lambda / \mu_2$, $b_i = \sum_{k=1}^{\infty} k^i a_k$ $(i \geq 1)$, $\rho_1 = \alpha_1 b_1$, $\rho_2 = \alpha_2 b_1$.

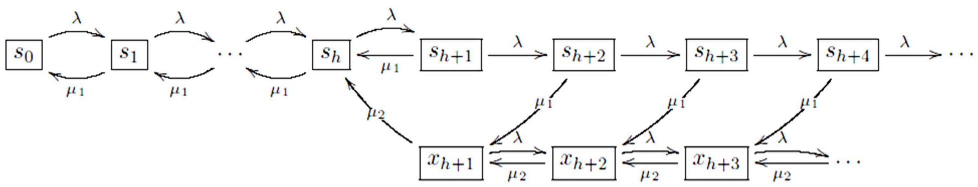

Fig. 1. States graph of $\mathrm{M}^\theta/\mathrm{M}_1/1$ queue in case $a_1 = 1$

We introduce the following numbering of states of the queueing system (see Fig. 1): state s_0 means that the system is free; state s_i $(i \geq 1)$ means the presence in the system of i customers and that service is performed in the main mode; x_i $(i \geq h+1)$ means the presence in the system of i customers and that a service is performed with the intensity μ_2.

Denote by $p_i(t)$ ($q_i(t)$) the probability of the system at time t remaining state (s_i x_i). Assuming that the process of change of system states is ergodic (the conditions of ergodity are offered below), i.e., there exist limits $p_i = \lim_{t\to\infty} p_i(t)$ $(i \geq 0)$, $q_i = \lim_{t\to\infty} q_i(t)$ $(i \geq h+1)$, we write the system of equations for finding stationary probabilities p_i and q_i

$$-\lambda p_0 + \mu_1 p_1 = 0; \quad -(\lambda + \mu_1) p_i + \lambda \sum_{k=0}^{i-1} p_k a_{i-k} + \mu_1 p_{i+1} = 0 \qquad (i = \overline{1, h-1}) \quad (1)$$

$$-(\lambda+\mu_1)p_h+\lambda\sum_{k=0}^{h-1}p_k a_{h-k}+\mu_1 p_{h+1}+\mu_2 q_{h+1}=0 \tag{2}$$

$$-(\lambda+\mu_1)p_i+\lambda\sum_{k=0}^{i-1}p_k a_{i-k}=0 \quad (i\ge h+1) \tag{3}$$

$$-(\lambda+\mu_2)q_{h+1}+\mu_1 p_{h+2}+\mu_2 q_{h+2}=0 \tag{4}$$

$$-(\lambda+\mu_2)q_i+\lambda\sum_{k=h+1}^{i-1}q_k a_{i-k}+\mu_1 p_{i+1}+\mu_2 q_{i+1}=0 \quad (i\ge h+2) \tag{5}$$

$$\sum_{k=0}^{\infty}p_k+\sum_{k=h+1}^{\infty}q_k=1 \tag{6}$$

Let

$$A_i=1-\sum_{k=1}^{i}a_k \ \ (i\ge 0),\ \ A_0=1;\ \ p_i=p_0\tilde{p}_i \ \ (i\ge 0),\ \ \tilde{p}_0=1;\ \ q_i=p_0\tilde{q}_i \ \ (i\ge h+1)$$
$$P_h=\sum_{i=1}^{h}\tilde{p}_i;\ \ Q_h=A_h+\sum_{i=1}^{h}\tilde{p}_i A_{h-i};\ \ L_h=\sum_{i=1}^{h}i\tilde{p}_i;\ \ L_h^{(2)}=\sum_{i=1}^{h}i^2\tilde{p}_i \tag{7}$$

Theorem 1. *If* $b_1<\infty$ *and* $\rho_2<1$, *then stationary probabilities* p_i $(i\ge 0)$ *and* q_i $(i\ge h+1)$ *exist and can be determined by recurrence relations*

$$p_{i+1}=\alpha_1\sum_{k=0}^{i}p_k A_{i-k} \qquad (i=\overline{0,h-1}) \tag{8}$$

$$q_i=\alpha_2\left(\sum_{k=0}^{i}p_k A_{i-k}+\sum_{k=h+1}^{i-1}q_k A_{i-1-k}\right) \qquad (i\ge h+1) \tag{9}$$

$$p_i=\frac{\lambda}{\lambda+\mu_1}\sum_{k=0}^{i-1}p_k a_{i-k} \qquad (i\ge h+1) \tag{10}$$

$$p_0=\frac{\mu_1(\mu_2-\lambda b_1)}{\mu_1\mu_2+(\mu_2-\mu_1)(\mu_1 P_h+\lambda Q_h)} \tag{11}$$

Proof. We get relations (8) by summing successively i ($i\ge 1$) equations of system (1), and equalities (10) follow directly from equations (3).

In order to obtain the first equality in (9), we add equation (2) and then the first equation in (3) to the sum of all the equations in (1) (i.e., to the last equality in (8)). To obtain the second equality in (9), we add the first equality in (9) and equation (4) to the second equation in (3). Summing the third equation in (3), the second equality in (9) and the first equation in (5), we obtain the third equality in (9) and so on.

To find p_0, we sum over i ($i \geq 1$) both sides of the relations

$$\mu_1 p_i = \lambda \sum_{k=0}^{i-1} p_k A_{i-1-k} \ \ (i = \overline{1,h}); \quad \mu_1 p_i + \mu_2 q_i = \lambda \sum_{k=0}^{i-1} p_k A_{i-1-k} + \lambda \sum_{k=h+1}^{i-1} q_k A_{i-1-k} \ \ (i \geq h+1)$$

and, using normalization condition (6), we get

$$\sum_{i=0}^{\infty}\sum_{k=0}^{i} p_k A_{i-k} + \sum_{i=h+1}^{\infty}\sum_{k=h+1}^{i} q_k A_{i-k} = (\sum_{i=0}^{\infty} p_i + \sum_{i=h+1}^{\infty} q_i)\sum_{k=0}^{\infty} A_k = \sum_{k=0}^{\infty} A_k = \sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty} a_j = \sum_{j=1}^{\infty} j a_j = b_1$$

Then

$$\lambda b_1 = \mu_1 \sum_{i=1}^{\infty} p_i + \mu_2 \sum_{i=h+1}^{\infty} q_i = \mu_1 (1 - p_0 - \sum_{i=h+1}^{\infty} q_i) + \mu_2 \sum_{i=h+1}^{\infty} q_i =$$
$$= \mu_1 (1 - p_0) + (\mu_2 - \mu_1) \sum_{i=h+1}^{\infty} q_i \tag{12}$$

Summing over i ($i \geq h+1$), both sides of relations (9) and using equalities (8) we deduce that

$$\sum_{i=h+1}^{\infty} q_i = \alpha_2 \sum_{i=h+1}^{\infty} (\sum_{k=0}^{i} p_k A_{i-k} + \sum_{k=h+1}^{i-1} q_k A_{i-1-k}) = \alpha_2 (b_1 - \sum_{i=0}^{h}\sum_{k=0}^{i} p_k A_{i-k}) =$$
$$= b_1 \alpha_2 - \frac{\mu_1}{\mu_2} \sum_{i=1}^{h} p_i - \alpha_2 \sum_{k=0}^{h} p_k A_{h-k} \tag{13}$$

Substituting relation (13) into (12) and using notations (7), we obtain solution p_0 to equation (12) of form (11). Formula (11) gives positive values of p_0, provided only that $\mu_2 > \lambda b_1$, i.e., $\rho_2 < 1$. The proof of the theorem is now complete.

Recurrence relations (8)-(11) serve to determine $\tilde{p}_i$ ($i \geq 1$), $\tilde{q}_i$ ($i \geq h+1$), P_h, Q_h, L_h, $L_h^{(2)}$ and thus, stationary probabilities p_i ($i \geq 0$) and q_i ($i \geq h+1$) for every fixed value h.

Consider the generating functions

$$P(z) = \sum_{i=0}^{\infty} p_i z^i; \qquad Q(z) = \sum_{i=h+1}^{\infty} q_i z^i; \qquad \tilde{P}(z) = P(z) + Q(z); \qquad A(z) = \sum_{i=1}^{\infty} a_i z^i .$$

Theorem 2. *Generating function* $\tilde{P}(z)$ *can be represented as*

$$\tilde{P}(z) = \frac{1}{(\mu_1 + \lambda(1 - A(z)))(\mu_2 (1 - z) - \lambda z (1 - A(z)))}((\mu_2 (1 - z) - \lambda z (1 - A(z)) - \mu_1) \times$$
$$\times (\mu_1 \sum_{i=1}^{h+1} p_i z^{i-1} + + \mu_1 p_0 + \mu_2 q_{h+1} z^h) + (\mu_1 + \lambda(1 - A(z)))(\mu_1 \sum_{i=0}^{h+1} p_i z^i + \mu_2 q_{h+1} z^{h+1})) \tag{14}$$

Proof. Multiplying the *i*-th equation of system (1)-(3) by z^i $(i \ge 0)$ and summing them, we get

$$-(\lambda+\mu_1)P(z)+\lambda P(z)A(z)+\mu_1 p_0+\mu_1\sum_{i=1}^{h+1} p_i z^{i-1}+\mu_2 q_{h+1}z^h=0$$

Hence,

$$P(z)=\frac{1}{\mu_1+\lambda(1-A(z))}\left(\mu_1\sum_{i=1}^{h+1} p_i z^{i-1}+\mu_2 q_{h+1}z^h+\mu_1 p_0\right) \quad (15)$$

Now we multiply the *i*-th equation of system (4)-(5) by z^i and sum over i $(i \ge h+1)$. We obtain

$$(\mu_2(1-z)-\lambda z)Q(z)+\mu_1\left(P(z)-\sum_{i=0}^{h+1} p_i z^i\right)-\mu_2 q_{h+1}z^{h+1}+\lambda z Q(z)A(z)=0$$

From this equation using equality (15), we first find $Q(z)$ and then we find $\widetilde{P}(z)$ of form (14). This completes the proof.

By differentiating generating function $\widetilde{P}(z)$ at point $z=1$ (an appropriate number of times) it is possible to calculate the moments of the number of customers in the system. In particular, denoting by $\overline{L}$ the stationary average number of customers in the system, we find that

$$\overline{L}=\sum_{i=1}^{\infty} i p_i+\sum_{i=h+1}^{\infty} i q_i=\widetilde{P}'(1); \qquad \widetilde{P}''(1)=\sum_{i=2}^{\infty} i(i-1)p_i+\sum_{i=h+1}^{\infty} i(i-1)q_i \quad (16)$$

Denote by $N(F)(z)$ and $D(F)(z)$ the numerator and denominator in the expression for some function $F(z)$, respectively. Let us simplify the process of finding the derivatives of function $\widetilde{P}(z)$.

Lemma 1. *For function* $\widetilde{P}(z)$ *defined by* (14), *the following equalities hold:*

$$\widetilde{P}'(1)=\frac{N''(\widetilde{P}')(1)}{D''(\widetilde{P}')(1)}=\frac{D'(\widetilde{P})(1)(N''(\widetilde{P})(1)-D''(\widetilde{P})(1))}{D''(\widetilde{P}')(1)} \quad (17)$$

$$\widetilde{P}''(1)=\frac{N^{(IV)}(\widetilde{P}'')(1)}{D^{(IV)}(\widetilde{P}'')(1)};\ N^{(IV)}(\widetilde{P}'')(1)=2D''(\widetilde{P}')(1)(2D'(\widetilde{P})(1)(N'''(\widetilde{P})(1)- -D'''(\widetilde{P})(1))-\widetilde{P}'(1)D'''(\widetilde{P}')(1)) \quad (18)$$

Proof. Since $A(1)=\sum_{i=1}^{\infty}a_i=1$, it is clear that for function $\widetilde{P}(z)$ defined by (14), the equalities hold

$$\begin{aligned}&N(\widetilde{P})(1)=D(\widetilde{P})(1)=N(\widetilde{P}')(1)=D(\widetilde{P}')(1)=N'(\widetilde{P}')(1)=D'(\widetilde{P}')(1)=0\\&D''(\widetilde{P}')(1)\neq 0\end{aligned}\qquad(19)$$

Using equality $\widetilde{P}(1)=1$ as well as L'Hôpital's rule and the representation of function $\widetilde{P}(z)$, we get

$$\begin{aligned}&N'(\widetilde{P})(1)=D'(\widetilde{P})(1)\neq 0;\quad D'(\widetilde{P}'')(1)=D''(\widetilde{P}'')(1)=D'''(\widetilde{P}'')(1)=0;\\&D^{(IV)}(\widetilde{P}'')(1)\neq 0\end{aligned}\qquad(20)$$

From (19) and (20), the first parts of formulas (17) and (18) follow respectively. Since

$$\begin{aligned}&N(\widetilde{P}')(z)=N'(\widetilde{P})(z)\cdot D(\widetilde{P})(z)-N(\widetilde{P})(z)\cdot D'(\widetilde{P})(z);\\&N'(\widetilde{P}')(z)=N''(\widetilde{P})(z)\cdot D(\widetilde{P})(z)-N(\widetilde{P})(z)\cdot D''(\widetilde{P})(z);\\&N''(\widetilde{P}')(z)=N'''(\widetilde{P})(z)\cdot D(\widetilde{P})(z)+N''(\widetilde{P})(z)\cdot D'(\widetilde{P})(z)-\\&-N'(\widetilde{P})(z)\cdot D''(\widetilde{P})(z)-N(\widetilde{P})(z)\cdot D'''(\widetilde{P})(z)\end{aligned}$$

in view of (19) and the first formula in (20), the following equality

$$N''(\widetilde{P}')(1)=D'(\widetilde{P})(1)(N''(\widetilde{P})(1)-D''(\widetilde{P})(1))$$

holds. By differentiating both sides of the equality

$$N(\widetilde{P}'')(z)=N'(\widetilde{P}')(z)\cdot D(\widetilde{P}')(z)-N(\widetilde{P}')(z)\cdot D'(\widetilde{P}')(z)$$

we obtain the formula

$$\begin{aligned}&N^{(IV)}(\widetilde{P}'')(z)=N^{(V)}(\widetilde{P}')(z)\cdot D(\widetilde{P}')(z)+3N^{(IV)}(\widetilde{P}')(z)\cdot D'(\widetilde{P}')(z)+\\&+2N'''(\widetilde{P}')(z)\cdot D''(\widetilde{P}')(z)-N(\widetilde{P}')(z)\cdot D^{(V)}(\widetilde{P}')(z)-\\&-3N'(\widetilde{P}')(z)\cdot D^{(IV)}(\widetilde{P}')(z)-2N''(\widetilde{P}')(z)\cdot D'''(\widetilde{P}')(z)\end{aligned}$$

which in view of relations (19) and the equality

$$N''(\widetilde{P}')(1)=\widetilde{P}'(1)\cdot D''(\widetilde{P}')(1)$$

which follows from the first formula in (17), at point $z=1$, it can be reduced to

$$N^{(IV)}(\widetilde{P}'')(1)=2D''(\widetilde{P}')(1)(N'''(\widetilde{P}')(1)-\widetilde{P}'(1)\cdot D'''(\widetilde{P}')(1))\qquad(21)$$

By using the formula

$$N'''(\tilde{P}')(z) = N^{(IV)}(\tilde{P})(z)\cdot D(\tilde{P})(z) + 2N'''(\tilde{P})(z)\cdot D'(\tilde{P})(z) - 2N'(\tilde{P})(z)\cdot D'''(\tilde{P})(z) - $$
$$-N(\tilde{P})(z)\cdot D^{(IV)}(\tilde{P})(z)$$

and the first relations in (19) and (20), we get

$$N'''(\tilde{P}')(1) = 2D'(\tilde{P})(1)(N'''(\tilde{P})(1) - D'''(\tilde{P})(1))$$

Substituting the expression for $N'''(\tilde{P}')(1)$ into (21), we establish a second formula in (18) and the lemma is proved.

Lemma 2. *The first and the second derivatives of function* $\tilde{P}(z)$ *at point* $z=1$ *are determined as follows:*

$$\tilde{P}'(1) = \frac{1}{2\mu_1(\mu_2-\lambda b_1)}(2\mu_1(\mu_2-\mu_1)\sum_{i=2}^{h+1}(i-1)p_i + 2\mu_2(\mu_2-\mu_1)hq_{h+1} - 2\lambda b_1\mu_1 p_0 + \lambda\mu_1(b_1+b_2) + 2\lambda b_1(\mu_2-\lambda b_1)) \quad (22)$$

$$\tilde{P}''(1) = \frac{1}{3\mu_1(\mu_2-\lambda b_1)}(3\mu_1(\mu_2-\mu_1)\sum_{i=3}^{h+1}(i-1)(i-2)p_i + 3\mu_2(\mu_2-\mu_1)h(h-1)q_{h+1} - 3\lambda\mu_1(b_2-b_1)p_0 + \lambda\mu_1(b_3-b_1) + 3\lambda\mu_2(b_2-b_1) - 6\lambda^2 b_1 b_2 + 3\lambda\tilde{P}'(1)(\mu_1(b_1+b_2) + 2b_1(\mu_2-\lambda b_1))) \quad (23)$$

Proof. To calculate derivatives $\tilde{P}'(1)$ and $\tilde{P}''(1)$, we use formulas (17) and (18), and take into consideration the equalities

$$A'(1) = \sum_{i=1}^{\infty} i a_i = b_1; \quad A''(1) = \sum_{i=2}^{\infty} i(i-1)a_i = b_2 - b_1$$
$$A'''(1) = \sum_{i=3}^{\infty} i(i-1)(i-2)a_i = b_3 - 3b_2 + 2b_1$$

The lemma is proved.

Theorem 3. *If* $b_i < \infty$ $(i=1,2)$ *and* $\rho_2 < 1$, *then the stationary average number of customers in an* $M^{\theta}/M_1/1$ *queue is finite and it is determined as follows:*

$$\bar{L} = \frac{1}{2\mu_1(\mu_2-\lambda b_1)}(\lambda\mu_1(b_1+b_2) + 2\lambda b_1(\mu_2-\lambda b_1) + 2p_0(\mu_1(\mu_2-\mu_1)(L_h - P_h + h\tilde{p}_{h+1}) + $$
$$+\mu_2(\mu_2-\mu_1)h\tilde{q}_{h+1} - \lambda b_1\mu_1)) \quad (24)$$

and the stationary average queue length as well as the stationary average waiting time are determined by

$$MQ = \overline{L} - 1 + p_0 = \overline{L} - \frac{(\mu_2 - \mu_1)(\mu_1 P_h + \lambda Q_h) + \lambda \mu_1 b_1}{\mu_1 \mu_2 + (\mu_2 - \mu_1)(\mu_1 P_h + \lambda Q_h)}; \qquad Mw = \frac{MQ}{\lambda b_1} \quad (25)$$

Proof. From the first formula in (16) and equality (22) using notations (7), we obtain (24). The first formula in (25) follows from the obvious equalities

$$MQ = \sum_{i=2}^{\infty}(i-1)p_i + \sum_{i=h+1}^{\infty}(i-1)q_i = \sum_{i=1}^{\infty} ip_i + \sum_{i=h+1}^{\infty} iq_i - \sum_{i=1}^{\infty} p_i - \sum_{i=h+1}^{\infty} q_i = \overline{L} - (1 - p_0)$$

and the second one is the consequence of Little's formula for systems with grouped arrivals of customers. This completes the proof.

Theorem 4. *If* $b_i < \infty$ $(i = 1,2,3)$ *and* $\rho_2 < 1$, *then the variance of the stationary average queue length in the* $\mathrm{M}^{\theta}/\mathrm{M}_1/1$ *queue is finite and it can be determined as follows:*

$$\begin{aligned}\mathbf{D}Q = \widetilde{P}''(1) - MQ - (MQ)^2 = \frac{1}{3\mu_1(\mu_2 - \lambda b_1)}(\lambda\mu_1(b_3 - b_1) + \\ + 3\lambda\mu_2(b_2 - b_1) - 6\lambda^2 b_1 b_2 + 3\lambda\overline{L}(\mu_1(b_1 + b_2) + 2b_1(\mu_2 - \lambda b_1)) + \\ 3p_0(\mu_1(\mu_2 - \mu_1)(L_h^{(2)} - 3L_h + 2P_h + h(h-1)\widetilde{p}_{h+1}) + \\ +\mu_2(\mu_2 - \mu_1)h(h-1)\widetilde{q}_{h+1} - \lambda\mu_1(b_2 - b_1))) - MQ - (MQ)^2\end{aligned} \quad (26)$$

Proof. In view of the second formula in (16), we obtain

$$\begin{aligned}MQ^2 = \sum_{i=2}^{\infty}(i-1)^2 p_i + \sum_{i=h+1}^{\infty}(i-1)^2 q_i = \\ = \widetilde{P}''(1) - \sum_{i=2}^{\infty}(i-1)p_i - \sum_{i=h+1}^{\infty}(i-1)q_i = \widetilde{P}''(1) - MQ\end{aligned}$$

Using expression (23) for $\widetilde{P}''(1)$ and notations (7), we arrive at formula (26) for $\mathbf{D}Q$. This completes the proof.

Now we investigate the limit performance capabilities of an $\mathrm{M}^{\theta}/\mathrm{M}_1/1$ queue as the intensity of service of post threshold mode μ_2 increases infinitely. The next theorem follows directly from equalities (11), (24) and (25).

Theorem 5. *Let the conditions of Theorem 3 hold. Then*

$$\lim_{\mu_2\to\infty} p_0(\mu_2) = \frac{\mu_1}{\mu_1(1+P_h)+\lambda Q_h}$$

$$\lim_{\mu_2\to\infty} \bar{L}(\mu_2) = \frac{1}{\mu_1}\left(\lambda b_1 + \frac{\mu_1(\mu_1(L_h - P_h + h\tilde{p}_{h+1}) + \mu_2 h\tilde{q}_{h+1})}{\mu_1(1+P_h)+\lambda Q_h}\right)$$

$$\lim_{\mu_2\to\infty} MQ(\mu_2) = \frac{1}{\mu_1}\left(\lambda b_1 + \frac{\mu_1(\mu_1(L_h - P_h + h\tilde{p}_{h+1}) + \mu_2 h\tilde{q}_{h+1})}{\mu_1(1+P_h)+\lambda Q_h}\right) - \frac{\mu_1 P_h + \lambda Q_h}{\mu_1(1+P_h)+\lambda Q_h}$$

If the minimum number of incoming customers in the group is comparable to threshold value h, then the stationary characteristics of the $M^{\theta}/M_1/1$ queue can be found in an explicit form.

Theorem 6. *If* $b_i < \infty$ $(i = 1,2,3)$, $\rho_2 < 1$ *and the following conditions hold:*

$$a_k = 0 \quad (k = \overline{1,h-2}); \qquad a_k \ge 0 \quad (k \ge h-1) \tag{27}$$

then

$$p_0 = \frac{\mu_1(\mu_2 - \lambda b_1)}{\mu_1\mu_2 + (\mu_2 - \mu_1)(\mu_1 P_h + \lambda(A_h + P_h - \alpha_1 a_{h-1}))}$$
$$p_i = \alpha_1 p_0 (1+\alpha_1)^{i-1} \ (i = \overline{1,h-1}) \tag{28}$$

and the stationary characteristics of the $M^{\theta}/M_1/1$ *queue are finite and they are determined by formulas* (24)-(26), *where*

$$P_h = (1+\alpha_1)^h - 1 - \alpha_1 a_{h-1}; \qquad L_h = \frac{1}{\alpha_1}(1 + (\alpha_1 h - 1)(1+\alpha_1)^h - \alpha_1^2 h a_{h-1});$$
$$L_h^{(2)} = L_h + \frac{1}{\alpha_1^2}((1+\alpha_1)(h(h-1)(1+\alpha_1)^{h+1} - 2(h^2-1)(1+\alpha_1)^h + \tag{29}$$
$$+h(h+1)(1+\alpha_1)^{h-1} - 2) - \alpha_1^3 h(h-1)a_{h-1})$$

$$Q_h = P_h + A_h - \alpha_1 a_{h-1}; \qquad \tilde{p}_{h+1} = \frac{\lambda}{\lambda+\mu_1}(a_{h+1} + \alpha_1 a_h + \alpha_1(1+\alpha_1)a_{h-1});$$
$$\tilde{q}_{h+1} = \alpha_2(A_{h+1} + \alpha_1 A_h + \alpha_1(1+\alpha_1)A_{h-1} + (1+\alpha_1)^h - (1+\alpha_1)^2 - \alpha_1 a_{h-1} + \tilde{p}_{h+1}) \tag{30}$$

Proof. If conditions (27) are satisfied, then from relations (7) it follows that

$$A_k = 1 \quad (k = \overline{1,h-2}); \qquad A_k = 1 - \sum_{i=h-1}^{k} a_i \qquad (k \ge h-1) \tag{31}$$

In view of (31), by using formula (8) we obtain

$$p_{i+1} = \alpha_1 \sum_{k=0}^{i} p_k = \alpha_1 (1+\alpha_1)^i p_0 \qquad (i = \overline{0, h-2});$$

$$p_h = \alpha_1 (p_0 A_{h-1} + \sum_{k=1}^{h-1} p_k) = \alpha_1 ((1+\alpha_1)^{h-1} - a_{h-1}) p_0$$

This leads to the expression for P_h in (29). The expressions in (29) for L_h and $L_h^{(2)}$ are obtained using the formulas

$$\sum_{k=1}^{h} k x^k = \frac{x(1+hx^{h+1}-(h+1)x^h)}{(x-1)^2}; \quad \sum_{k=2}^{h} k(k-1)x^k = \frac{x^2}{(x-1)^3}(h(h-1)x^{h+1} -$$
$$-2(h^2-1)x^h + h(h+1)x^{h-1} - 2)$$

Formulas (30) follow from (9) and (10) with the use of equalities (31) and expressions (28) for p_i $(i = \overline{1, h-1})$. Substituting expression (30) for Q_h into formula (11), we obtain equality (28) for p_0. The proof of the theorem is now complete.

3. Examples of calculating stationary characteristics

We calculate the stationary characteristics of an $\mathrm{M}^{\theta}/\mathrm{M}_1/1$ queue in the case that customers can arrive only one or two at a time, i.e., $a_1 + a_2 = 1$. Let $\lambda = 2$, $\mu_1 = 1$, $\mu_2 = 4$, $a_1 = 0.75$, $a_2 = 0.25$ (data 1). Then $\alpha_1 = 2$; $\alpha_2 = 0.5$; $b_1 = 1.25$; $b_2 = 1.75$; $b_3 = 2.75$; $\rho_1 = 2.5$; $\rho_2 = 0.625$; $A_1 = 0.25$; $A_i = 0$ $(i \geq 2)$.

Table 1

Stationary characteristics of M^{θ}/M/1 queue (for data 1)

h	1	2	3	4	5
$\mathbf{M}Q$	4.032	4.855	5.768	6.725	7.704
$\mathbf{M}Q$ (GPSS)	4.039	4.850	5.779	6.705	7.700
$\sigma(Q)$	4.006	4.097	4.153	4.187	4.207
$\sigma(Q)$ (GPSS)	3.986	4.070	4.147	4.156	4.210
$\mathbf{M}w$	1.613	1.942	2.307	2.690	3.082
$\mathbf{M}w$ (GPSS)	1.615	1.939	2.309	2.682	3.082

Denote by $\sigma(Q)$ a standard deviation of the stationary queue length, i.e., $\sigma(Q)=\sqrt{\mathbf{D}Q}$. Table 1 contains the values of the stationary characteristics of an $M^{\theta}/M_1/1$ queue that are calculated for different threshold values h. In this table, for comparison, the values of these stationary characteristics obtained by the general purpose simulation system (GPSS World) [7, 8] are also written for the value of simulating time $t=5\cdot 10^5$.

Table 2

Stationary characteristics of $M^{\theta}/M/1$ queue (for data 2-5)

Data number	2	3	4	5
$\mathbf{M}Q$	10,906	14,503	27,024	53,733
$\mathbf{M}Q$ (GPSS)	10,907	14,665	27,315	53,881
$\mathbf{M}w$	1,818	2,231	3,753	7,070
$\mathbf{M}w$ (GPSS)	1,821	2,253	3,800	7,095

Assume that for an $M^{\theta}/M_1/1$ queue, conditions (27) hold. Let $\lambda=2$, $\mu_1=2$, $\mu_2=8$, $h=4$, $a_k=0$ $(k=1,2)$. Consider such cases as: $a_3=1$, $a_k=0$ $(k\ge 4)$ (data 2); $a_3=0,75$; $a_4=0,25$; $a_k=0$ $(k\ge 5)$ (data 3); $a_3=0,6$; $a_4=a_5=0,2$; $a_k=0$ $(k\ge 6)$ (data 4); $a_3=0,5$; $a_4=0,3$; $a_5=a_6=0,1$; $a_k=0$ $(k\ge 7)$ (data 5). The comparison of the stationary characteristics, calculated for data 2-5 with ones obtained with the assistance of GPSS World for the value of simulating time $t=2\cdot 10^5$ is exhibited in Table 2.

Let conditions (27) hold, and the range of the number of incoming customers in the group is a countable set described by the geometrical distribution: $a_{h-1}=p$; $a_{h+k}=pq^{k+1}$ $(k\ge 0)$, where $0<p<1$, $q=1-p$. Then stationary characteristics $\overline{L}$ and $\mathbf{M}Q$ can be found by using explicit formulas (24), (25), (29), (30), where

$$A_{h-1}=q,\quad A_h=1-p(q+1),\quad A_{h+1}=1-p(q^2+q+1)$$

$$b_1=\frac{p^2(h(q+1)-1)+hp+1}{p};\qquad b_2=p((h-1)^2+h^2q)+h^2+\frac{2hp+q+1}{p^2}$$

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