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THERMALLY INDUCED VIBRATION OF AN ANNULAR PLATE

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Abstract. The exact solution to a problem of the thermally induced vibration of a homogeneous annular plate is presented. The considered plate is subjected to the activity of a point heat source, which moves with a constant angular velocity on the plate surface along a trajectory. The thermal moment is derived on the basis of a temperature field in the plate. The solution to the vibration problem is obtained by using Green's function method.

Introduction

The thermally induced vibration of beams and plates is of great interest to engineers due to its practical importance in mechanical, chemical, aeronautical and nuclear power industries. Several authors have studied the problem of the thermally induced vibration of plates [1-7].

In paper [1], the equation of a thermally excited vibration of a circular plate is derived. The plate forced by a temperature field varying harmonically in time was considered. The heat conduction problem was solved by means of the finite Hankel transformation and the solution was found in the form of a series. In paper [2], the thermally induced vibrations of simply supported and clamped circular plates were studied. In this analysis, it is assumed that the distribution of temperature is linear through the thickness and along the radius. To solve this problem, the authors used an analytical method (the method of separation of variables) and a numerical method (the finite element method). The non-linear response of a thermally loaded isotropic plate was investigated by Haider, Arafat and Nayfeh [3]. The plate was excited externally by a harmonic force near the primary resonance. The authors considered the in-plane thermal load to be axisymmetric. In paper [4], the authors investigated an inverse thermoelastic problem in a thin isotropic circular plate. The authors determined the temperature distribution and thermal deflection on the curved surface of the plate by employing an integral transform. The results were obtained in terms of series of Bessel's functions. The thermally induced vibration of a circular and annular plate is presented in paper [7]. The plate was subjected to a sinusoidally varying heat flux on one surface and the other is thermally insulated. Applying the theory to circular and annular plates, the deflection, the stress distribution and the frequency response of the plates were calculated numerically. In paper [8], the problem of the thermally induced vibration of a circular plate was solved by using Green's function method.

In this paper, an analytical solution to the problem of the thermally induced vibration of an annular plate is presented. The thermal moment caused by the temperature distribution on the thin annular plate is determined and displacements of the plate induced by the thermal moment are analyzed theoretically. The solution to the problem is obtained by using a time-dependent Green's function.

1. Heat conduction problem

An annular isotropic plate of uniform thickness *h* with inner radius *a* and outer radius *b* (Fig. 1) is considered. This plate is heated by a heat source which moves on the plate surface along a concentric circular trajectory at radius r_0 with constant angular velocity ω.

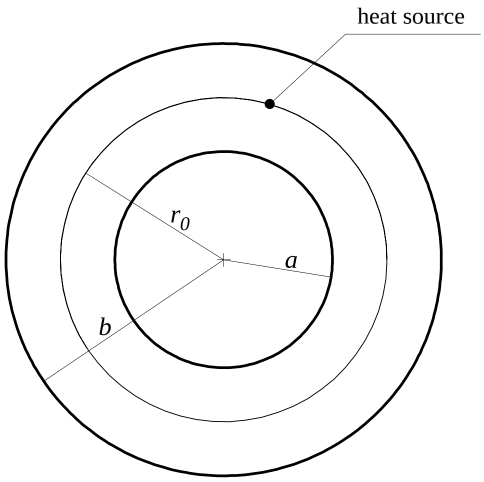

Fig. 1. Schema of annular plate with heat source

The temperature of the plate is governed by the heat conduction equation which in cylindrical coordinates is as follows

$$\nabla^2 T + \frac{\partial^2 T}{\partial z^2} + \frac{1}{k} q(r,\phi,z,t) = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1}$$

where: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$, $T(r,\phi,z,t)$ - temperature of the plate at point (r,ϕ,z) at time *t*, k - thermal conductivity, κ - thermal diffusivity and $q(r,\phi,z,t)$ represents the heat generation term. The heat generation term is assumed in the form:

$$q(r,\varphi,z,t) = \theta\,\delta(r - r_0)\,\delta(\varphi - \phi(t))\,\delta(z - h) \tag{2}$$

where θ characterises the stream of the heat, $\delta(\ )$ is the Dirac delta function, φ(t) is the function describing the movement of the heat source

$$\phi(t) = \omega\, t \tag{3}$$

An analytical form of the temperature distribution in the considered plate has been given in paper [9] as a solution of equation (1) with the following initial and boundary conditions:

$$T(r,\phi,z,0) = 0 \tag{4}$$

$$T\big|_{r=a} = T_1\,, \quad T\big|_{r=b} = T_2 \tag{5}$$

$$k\frac{\partial T}{\partial z}(r,\phi,h,t) = \alpha_0[T_0 - T(r,\phi,h,t)] \tag{6}$$

$$k\frac{\partial T}{\partial z}(r,\phi,0,t) = -\alpha_0[T_0 - T(r,\phi,0,t)] \tag{7}$$

where α_0 is the heat transfer coefficient, T_0 is the known temperature of the surrounding medium. The temperature for T_0 = 0 is expressed as (derivation is presented in paper [9])

$$T(r,\phi,z,t) = \frac{\theta\,\kappa}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{R_{mk}(\rho)\psi_n(h)}{Q_n\,\chi_{mk}}R_{mk}(r)\Psi_n(z)P_{mnk}(t,\theta) \tag{8}$$

where

$$R_{mk}(r) = Y_m(\gamma_{mk}a)J_m(\gamma_{mk}r) - J_m(\gamma_{mk}a)Y_m(\gamma_{mk}r)$$

$$\psi_n(z) = \beta_n\cos\beta_n z + \mu_0\sin\beta_n z,\quad n = 1,2,...$$

γ_{mk} are the roots of the equation

$$J_m(\gamma_{mk}a)Y_m(\gamma_{mk}b) - J_m(\gamma_{mk}b)Y_m(\gamma_{mk}a) = 0 \tag{9}$$

and β_n are the roots of the equation

$$2\mu_0\,\beta_n\cos\beta_n h - \left(\beta_n^2 - {\mu_0}^2\right)\sin\beta_n h = 0 \tag{10}$$

$$p_{mnk}(t,\varphi) = \frac{1}{\vartheta_{mnk}^2 + m^2\omega^2}\Big[\vartheta_{mnk}\cos m(\varphi - \omega t) - m\omega\sin m(\varphi - \omega t) - \left(\vartheta_{mnk}\cos m\varphi - m\omega\sin m\varphi\right)e^{-\vartheta_{mnk}t}\Big] \tag{11}$$

where $\vartheta_{mnk} = \kappa\left(\beta_n^2 + \gamma_{mk}^2\right)$.

$$Q_n == \frac{h}{2}\left(\beta_n^2 + \mu_0^2\right)\left(1 + \frac{\beta_n^2 + \mu_0{}^2}{2\mu_0 h \beta_n^2}\sin^2 \beta_n h\right) \tag{12}$$

$$\chi_{mk} = \phi_{mk}^{[1]} J_m^2\left(a\gamma_{mk}\right) + \phi_{mk}^{[2]} Y_m^2\left(a\gamma_{mk}\right) - 2\phi_{mk}^{[3]} J_m\left(a\gamma_{mk}\right) Y_m\left(a\gamma_{mk}\right) \tag{13}$$

$$\phi_{mk}^{[1]} = -\frac{a^2}{2}\left(Y_{m-1}^2\left(a\gamma_{mk}\right) + Y_m^2\left(a\gamma_{mk}\right)\right) + \frac{b^2}{2}\left(Y_{m-1}^2\left(b\gamma_{mk}\right) + Y_m^2\left(b\gamma_{mk}\right)\right)$$

$$+ \frac{m}{\gamma_{mk}}\left(a\, Y_{m-1}\left(a\gamma_{mk}\right) Y_m\left(a\gamma_{mk}\right) - b\, Y_{m-1}\left(b\gamma_{mk}\right) Y_m\left(b\gamma_{mk}\right)\right)$$

$$\phi_{mk}^{[2]} = -\frac{a^2}{2}\left(J_{m-1}^2\left(a\gamma_{mk}\right) + J_m^2\left(a\gamma_{mk}\right)\right) + \frac{b^2}{2}\left(J_{m-1}^2\left(b\gamma_{mk}\right) + J_m^2\left(b\gamma_{mk}\right)\right)$$

$$+ \frac{m}{\gamma_{mk}}\left(a\, J_{m-1}\left(a\gamma_{mk}\right) J_m\left(a\gamma_{mk}\right) - b\, J_{m-1}\left(b\gamma_{mk}\right) J_m\left(b\gamma_{mk}\right)\right)$$

$$\phi_{mnk}^{[3]} = \frac{1}{2\sqrt{\pi}}\left[b^2 G_{3,5}^{2,2}\left(\left(b\gamma_{mk}\right)^2 \middle| \begin{matrix} 0, 0.5, -0.5 \\ 0, m, -m, -1, -0.5 \end{matrix}\right) - a^2 G_{2,4}^{2,2}\left(\left(a\gamma_{mk}\right)^2 \middle| \begin{matrix} 0, 0.5, -0.5 \\ 0, m, -m, -1, -0.5 \end{matrix}\right)\right]$$

where $G_{p,q}^{m,n}\left(x \middle| \begin{matrix} a_1,\ldots,a_n, a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m, b_{m+1},\ldots,b_q \end{matrix}\right)$ is a G-Meijer function [10].

2. Problem of thermally induced vibration of annular plate

The thermally induced vibration of the considered plate is governed by the biharmonic differential equation [8]

$$D\nabla^4 w + \mu \frac{\partial^2 w}{\partial t^2} = -\nabla^2 M_T \tag{14}$$

where D is the flexural stiffness, μ is the mass per unit area of the plate, $w(r,\Phi,t)$ is the displacement of the middle surface of the plate at point (r,Φ) at time t, and M_T denotes the thermal moment. The thermal moment appears as a result of a temperature field in the plate and it is defined as [8]

$$M_T = \frac{\alpha E}{1-\nu}\int_0^h zT\left(r,\varphi,z,t\right) dz \tag{15}$$

The presented study deals with an annular plate with simply supported edges, which means that the following boundary conditions are satisfied

$$w = 0\,, \quad -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \phi^2}\right)\right] = 0 \qquad \text{on } r = a \text{ and } r = b \tag{16}$$

Moreover, the zero initial conditions are assumed

$$w = \frac{\partial w}{\partial t} = 0 \qquad \text{for } t = 0 \tag{17}$$

Substituting (8) for equation (15), we obtain the thermal moment in the form:

$$M_T(r,\phi,t) = \frac{8\kappa\alpha E\mu_0 \Theta}{(1-\nu)\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{B_n}{Q_n\,\chi_{mnk}} R_{mk}(\gamma_{mk}r_0)R_{mk}(\gamma_{mk}r)\Psi_n(h)\,P_{mnk}(t,\phi) \tag{18}$$

where

$$B_n = \frac{(1+h\mu_0)\left(\beta_n^2+\mu_0^2\right)\sin\beta_n h - 2\beta_n\mu_0}{2\beta_n^2\mu_0}$$

The solution to problem (14), (16), (17) in an analytical form is obtained by using the properties of Green's function, which is a solution of differential equation [11]

$$D\nabla^4 G + \mu\frac{\partial^2 G}{\partial t^2} = -\frac{\delta(r-\rho)\delta(\phi-\psi)\delta(t-\tau)}{r} \tag{19}$$

and satisfies the zero initial and homogeneous boundary conditions analogous to conditions (16), (17). The solution to vibration problem (14), (16), (17) can be expressed as

$$w(r,\phi,t) = \int_0^t\int_0^b\int_0^{2\pi}\nabla^2 M_T(\rho,\psi,\tau)G(r,\phi,t;\rho,\psi,\tau)\,d\psi\,d\rho\,d\tau \tag{20}$$

3. Green's function

The GF for the considered vibration problem may be written in the form of a series

$$G(r,\phi,t) = \sum_{m=-\infty}^{\infty} g_m(r,t)\cos m(\phi-\psi) \tag{21}$$

Substituting series (18) for equation (16) and using the expansion [8]

$$\delta(\phi-\psi)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\cos m(\phi-\psi) \tag{22}$$

the differential equation for the functions $g_m(r,t)$ is obtained

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}\right)^2 g_m+\frac{\mu}{D}\frac{\partial^2 g_m}{\partial t^2}=\frac{\delta(r-\rho)\,\delta(t-\tau)}{2\pi D\; r} \tag{23}$$

Next, using (18) in boundary and initial conditions (13), (14), we have

$$g_m(a,t)=0\,,\quad \left[\frac{\partial^2 g_m}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial g_m}{\partial r}-\frac{m^2}{r^2}g_m\right)\right]\Bigg|_{r=a}=0 \tag{24}$$

$$g_m(b,t)=0\,,\quad \left[\frac{\partial^2 g_m}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial g_m}{\partial r}-\frac{m^2}{r^2}g_m\right)\right]\Bigg|_{r=b}=0 \tag{25}$$

$$g_m(r,0)=0\,,\quad \left.\frac{\partial g_m}{\partial t}\right|_{t=0}=0 \tag{26}$$

The solution to initial-boundary problem (23)-(26) can be presented in the form:

$$g_m(r,t)=\sum_{n=1}^{\infty}Q_{mn}(r)\,\Gamma_{mn}(t) \tag{27}$$

where $Q_{mn}(r)$ are the eigenfunctions of the following boundary problem:

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}\right]^2 Q_{mn}(r)-\lambda_{mn}^4 Q_{mn}(r)=0 \tag{28}$$

$$Q_{mn}(a)=0\,,\quad \left[\frac{\partial^2 Q_{mn}}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial Q_{mn}}{\partial r}-\frac{m^2}{r^2}Q_{mn}\right)\right]\Bigg|_{r=a}=0 \tag{29}$$

$$Q_{mn}(b)=0\,,\quad \left[\frac{\partial^2 Q_{mn}}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial Q_{mn}}{\partial r}-\frac{m^2}{r^2}Q_{mn}\right)\right]\Bigg|_{r=b}=0 \tag{30}$$

The general solution of differential equation (28) can be written in the form:

$$Q_{mn}(r)=C_1 J_m(\lambda_{mn}r)+C_2 Y_m(\lambda_{mn}r)+C_3 I_m(\lambda_{mn}r)+C_4 K_m(\lambda_{mn}r) \tag{31}$$

where J_m, Y_m are the Bessel functions of order *m*, and I_m, K_m are the modified Bessel functions of order *m*. Substituting function (31) for boundary conditions (29), (30), we obtain a system of homogeneous equations

$$\sum_{j=1}^{4} a_{ij} C_j = 0, \quad i = 1, \ldots, 4 \tag{32}$$

where:

$$A_{11} = J_m(\lambda_{mn} a), \quad A_{12} = Y_m(\lambda_{mn} a), \quad A_{13} = I_m(\lambda_{mn} a), \quad A_{14} = K_m(\lambda_{mn} a)$$

$$A_{21} = J_m(\lambda_{mn} b), \quad A_{22} = Y_m(\lambda_{mn} b), \quad A_{23} = I_m(\lambda_{mn} b), \quad A_{24} = K_m(\lambda_{mn} b)$$

$$A_{31} = \left(\frac{p_m}{a^2} - \lambda_{mn}^2\right) J_m(\lambda_{mn} a) + \frac{1-\nu}{a} \lambda_{mn} J_{m+1}(\lambda_{mn} a),$$

$$A_{32} = \left(\frac{p_m}{a^2} - \lambda_{mn}^2\right) Y_m(\lambda_{mn} a) + \frac{1-\nu}{a} \lambda_{mn} Y_{m+1}(\lambda_{mn} a)$$

$$A_{33} = \left(\frac{p_m}{a^2} + \lambda_{mn}^2\right) I_m(\lambda_{mn} a) - \frac{1-\nu}{a} \lambda_{mn} I_{m+1}(\lambda_{mn} a),$$

$$A_{34} = \left(\frac{p_m}{a^2} + \lambda_{mn}^2\right) K_m(\lambda_{mn} a) + \frac{1-\nu}{a} \lambda_{mn} K_{m+1}(\lambda_{mn} a)$$

$$A_{41} = \left(\frac{p_m}{b^2} - \lambda_{mn}^2\right) J_m(\lambda_{mn} b) + \frac{1-\nu}{b} \lambda_{mn} J_{m+1}(\lambda_{mn} b),$$

$$A_{42} = \left(\frac{p_m}{b^2} - \lambda_{mn}^2\right) Y_m(\lambda_{mn} b) + \frac{1-\nu}{b} \lambda_{mn} Y_{m+1}(\lambda_{mn} b)$$

$$A_{43} = \left(\frac{p_m}{b^2} + \lambda_{mn}^2\right) I_m(\lambda_{mn} b) - \frac{1-\nu}{b} \lambda_{mn} I_{m+1}(\lambda_{mn} b),$$

$$A_{44} = \left(\frac{p_m}{b^2} + \lambda_{mn}^2\right) K_m(\lambda_{mn} b) + \frac{1-\nu}{b} \lambda_{mn} K_{m+1}(\lambda_{mn} b)$$

where $p_m = m(m-1)(1-\nu)$.

A non-trivial solution of system (32) exists for these λ_{mn}, which satisfy equation

$$\det\left[A_{ij}\right] = 0 \tag{33}$$

This equation can be written in the form:

$$
\begin{aligned}
&\Psi_{IY}^{[m+1]}(a,b,\lambda)\Psi_{JK}^{[m]}(a,b,\lambda)-\Psi_{IK}^{[m+1]}(a,b,\lambda)\Psi_{JY}^{[m]}(a,b,\lambda)\\
&+\Psi_{IJ}^{[m+1]}(a,b,\lambda)\Psi_{KY}^{[m]}(a,b,\lambda)-\Psi_{KY}^{[m+1]}(a,b,\lambda)\Psi_{IJ}^{[m]}(a,b,\lambda)\\
&+\Psi_{JY}^{[m+1]}(a,b,\lambda)\Psi_{IK}^{[m]}(a,b,\lambda)-\Psi_{JK}^{[m+1]}(a,b,\lambda)\Psi_{IY}^{[m]}(a,b,\lambda)=0
\end{aligned}
\tag{34}
$$

where:

$$\Psi_{IJ}^{[m]}(a,b,\lambda)=I_m(a\lambda)J_m(b\lambda)-I_m(b\lambda)J_m(a\lambda)$$

$$\Psi_{IK}^{[m]}(a,b,\lambda)=I_m(a\lambda)K_m(b\lambda)-I_m(b\lambda)K_m(a\lambda)$$

$$\Psi_{IY}^{[m]}(a,b,\lambda)=I_m(a\lambda)Y_m(b\lambda)-I_m(b\lambda)Y_m(a\lambda)$$

$$\Psi_{JK}^{[m]}(a,b,\lambda)=J_m(a\lambda)K_m(b\lambda)-J_m(b\lambda)K_m(a\lambda)$$

$$\Psi_{JY}^{[m]}(a,b,\lambda)=J_m(a\lambda)Y_m(b\lambda)-J_m(b\lambda)Y_m(a\lambda)$$

$$\Psi_{KY}^{[m]}(a,b,\lambda)=K_m(a\lambda)Y_m(b\lambda)-K_m(b\lambda)Y_m(a\lambda)$$

The roots of equation (34) are determined numerically. The eigenfunctions corresponding to the roots are derived by using the solution of system (32) in equation (31). After transformations, the eigenfunctions can be presented in the form:

$$
\begin{aligned}
&Q_{mn}(r)=\frac{1}{a\,\lambda_{mn}}\left(-\Psi_{JY}^{[m]}(r,b,\lambda_{mn})+\frac{2}{\pi}\Psi_{IK}^{[m]}(r,b,\lambda)\right)\\
&-\overline{\Psi}_{IK}^{[m+1]}(a,b,\lambda)\Psi_{JY}^{[m]}(a,r,\lambda)-\Psi_{JY}^{[m+1]}(a,b,\lambda)\Psi_{IK}^{[m]}(a,r,\lambda)\\
&-\Psi_{JY}^{[m+1]}(a,b,\lambda)\Psi_{IK}^{[m]}(a,b,\lambda)+\Psi_{IK}^{[m+1]}(a,r,\lambda)\Psi_{JY}^{[m]}(a,b,\lambda)=0
\end{aligned}
\tag{35}
$$

Note that functions R_{mn} satisfy the orthogonality condition

$$\int_0^b r\,R_{mn}(r)\,R_{mn'}(r)\,dr=\begin{cases}0 & \text{for } n'\neq n\\ \chi_m(\lambda_{mn}) & \text{for } n'=n\end{cases}\tag{36}$$

where

$$
\begin{aligned}
\chi_m(\lambda)=\frac{b^2}{2\lambda}\Big[&J_m^2(\lambda)I_{m-1}(\lambda)(-\lambda I_{m-1}(\lambda)+2(m-1)J_m(\lambda))\\
&I_m^2(\lambda)J_{m-1}(\lambda)(\lambda J_{m-1}(\lambda)-2(m-1)J_m(\lambda))+2\lambda J_m^2(\lambda)I_m^2(\lambda)\Big]
\end{aligned}
\tag{37}
$$

Taking into account (23), (24) and using orthogonality condition (31) in equations (20) and (22), we obtain the differential equation

$$\frac{\partial^2 \Gamma_{mn}(t)}{\partial t^2} + \frac{D}{b^4 \mu} \lambda_{mn}^4 \Gamma_{mn}(t) = \frac{Q_{mn}(\rho)}{2\pi \mu \chi_m(\lambda_{mn})} \delta(t-\tau) \tag{38}$$

and initial conditions

$$\Gamma_{mn}(0) = 0, \quad \left. \frac{d\Gamma_{mn}}{dt} \right|_{t=0} = 0 \tag{39}$$

The solution to initial problem (33), (34) has the form

$$\Gamma_{mn}(t) = \frac{Q_{mn}(\rho)}{2\pi \mu \Omega_{mn} \chi_m(\lambda_{mn})} \sin \Omega_{mn}(t-\tau) H(t-\tau) \tag{40}$$

where $\Omega_{mn}^2 = \frac{D}{b^4 \mu} \lambda_{mn}^4$ and H denotes the Heaviside function.

Finally, on the basis of equations (18), (23) Green's function for the simply supported circular plate can be written in the following form

$$G(r,\phi,t;\rho,\Psi,\tau) = \frac{H(t-\tau)}{2\pi\mu} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{Q_{mn}(\rho)}{\Omega_{mn}\chi_m(\lambda_{mn})} Q_{mn}(r) \sin \Omega_{mn}(t-\tau) \cos m(\phi - \psi) \tag{41}$$

Summary

In this paper, the problem of the transverse vibration of an annular plate induced by a mobile heat source was considered. Formulation of the problem was based on the differential equations of heat conduction and transverse vibration of the plate, which were complemented by suitable initial and boundary conditions. The temperature distribution and transverse vibration of the annular plate in an analytical form were obtained by using the properties of Green's function.

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