VALUES OF FUNDAMENTAL SYMMETRIC POLYNOMIALS WITH NATURAL ARGUMENTS

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Abstract. In this paper we present explicit formulas for the values of fundamental symmetric polynomials with natural arguments.

Introduction

Symmetric polynomials are the polynomials of *n* (real or complex) variables whose values do not depend on the permutation of arguments. The simplest form of these polynomials (called fundamental polynomials) were presented in paper [1].

1. Fundamental symmetric and power polynomials

Fundamental symmetric polynomials appear when we calculate the product

$$
\begin{aligned}
(x-x_1)(x-x_2)\dots(x-x_n) &= x^n-(x_1+x_2+\dots+x_n)x^{n-1} \\
&+(x_1x_2+\dots+x_1x_n+x_2x_3+\dots+x_{n-1}x_n)x^{n-2} \\
&-(x_1x_2x_3+\dots+x_1x_2x_n+\dots+x_{n-2}x_{n-1}x_n)x^{n-3}+\dots \\
&+(-1)^n x_1x_2\dots x_n = x^n-\tau_1x^{n-1}+\tau_2x^{n-2}+\dots+(-1)^n\tau_n
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_1 &= x_1+x_2+\dots+x_n \\
\tau_2 &= x_1x_2+\dots+x_1x_n+x_2x_3+\dots+x_{n-1}x_n \\
&\dots\dots\dots\dots\dots\dots \\
\tau_n &= x_1x_2\dots x_n
\end{aligned}
\tag{1}
$$

Power symmetric polynomials appear when we calculate the expression

$$\left(x - x_1^k\right)\left(x - x_2^k\right) \dots \left(x - x_n^k\right) =$$
$$x^n - \left(x_1^k + x_2^k + \dots + x_n^k\right)x^{n-1} + \dots + (-1)^n x_1^k x_2^k \dots x_n^k$$

where

$$\sigma_k = x_1^k + x_2^k + \dots + x_n^k, \; k \geq 1 \tag{2}$$

Proposition 1.

Polynomials τ_k and σ_k are related by the Newton formula

$$\sigma_k - \sigma_{k-1} \cdot \tau_1 + \sigma_{k-2} \cdot \tau_2 - \dots + (-1)^{k-1} \cdot \sigma_1 \cdot \tau_{k-1} + (-1)^k \cdot \tau_k = 0 \;\; k \geq 1$$

The proof to this proposition is given in book [2].

2. Fundamental polynomials with natural arguments

The values of fundamental symmetric polynomials with natural arguments appear when we expand the polynomial

$$(x-1)(x-2) \dots (x-n) = x^n - (1 + 2 + \dots + n)x^{n-1}$$
$$+(1 \cdot 2 + \dots + 1 \cdot n + 2 \cdot 3 + \dots + (n-1) \cdot n)x^{n-2}$$
$$-(1 \cdot 2 \cdot 3 + \dots + (n-2) \cdot (n-1) \cdot n)x^{n-3} + \dots + (-1)^n \cdot 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$$

In this case we have

$$\tau_1 = 1 + 2 + \dots + n = \frac{1}{2} n \cdot (n+1)$$
$$\tau_2 = 1 \cdot 2 + \dots + 1 \cdot n + 2 \cdot 3 + \dots + (n-1) \cdot n$$
$$\tau_3 = 1 \cdot 2 \cdot 3 + \dots + (n-2) \cdot (n-1) \cdot n$$
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$$\tau_n = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-2) \cdot (n-1) \cdot n = n!$$

The values of power symmetric polynomials with natural arguments appear when we calculate the product

$$\left(x - 1^k\right)\left(x - 2^k\right) \dots \left(x - n^k\right)$$
$$= x^n - \left(1^k + 2^k + \dots + n^k\right)x^{n-1} + \dots + (-1)^n 1^k \cdot 2^k \cdot \dots, \cdot n^k$$

In this case

$$\sigma_k = 1^k + 2^k + \dots + n^k$$

3. Calculation of σ_k

Proposition 2.

A recursive formula for σ_k holds

$$\binom{k+1}{1}\cdot\sigma_1+\binom{k+1}{2}\cdot\sigma_2+\cdots+\binom{k+1}{k}\cdot\sigma_k=$$

$$=(n+1)\cdot n\cdot\left[n^{k-1}+\binom{k}{1}n^{k-2}+\cdots+\binom{k}{k-1}\right],\ k\geq 1 \qquad (3)$$

Proof.

To calculate the values of σ_k, we use the Newton binomial formula

$$(k+1)^p-k^p=\binom{p}{1}\cdot k^{p-1}+\binom{p}{2}\cdot k^{p-2}+\cdots+\binom{p}{p-1}\cdot k+1$$

Putting in this formula $k=1,2,\dots,n$, successively, we have

$k=1$: $\quad 2^p-1^p=\binom{p}{1}\cdot 1^{p-1}+\binom{p}{2}\cdot 1^{p-2}+\cdots+\binom{p}{p-1}\cdot 1+1$

$k=2$: $\quad 3^p-2^p=\binom{p}{1}\cdot 2^{p-1}+\binom{p}{2}\cdot 2^{p-2}+\cdots+\binom{p}{p-1}\cdot 2+1$

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$k=n$: $\quad (n+1)^p-n^p=\binom{p}{1}\cdot n^{p-1}+\binom{p}{2}\cdot n^{p-2}+\cdots+\binom{p}{p-1}\cdot n+1$

and summing both sides of these equations, we obtain

$$(n+1)^p-1=\binom{p}{1}\cdot\sigma_{p-1}+\binom{p}{2}\cdot\sigma_{p-2}+\cdots+\binom{p}{p-1}\cdot\sigma_1+n$$

This formula agrees with formula (3).

Proposition 3.

The values of σ_k can be calculated in another way, namely

$$\sigma_{k-1}=\frac{1}{k}\cdot\left[n^k+\binom{k}{1}\cdot n^{k-1}\cdot B^1+\binom{k}{2}\cdot n^{k-2}\cdot B^2+\binom{k}{3}\cdot n^{k-3}\cdot B^3+\cdots\right] \qquad (4)$$

where

$$B^1=\frac{1}{2},\qquad B^2=\frac{1}{6},\qquad B^3=0,\qquad B^4=\frac{-1}{16},\qquad B^5=0,\qquad \dots$$

are the Bernoulli numbers. The above formula is named the Faulhaber formula.

The proof of this formula can be found in book [3].

4. Calculation of τ_k

Using the determinant (see [2])

$$\tau_k = \frac{1}{k!}\begin{vmatrix} \sigma_1 & 1 & 0 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & 2 & 0 & \cdots & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \sigma_{k-4} & \cdots & k-1 \\ \sigma_k & \sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \cdots & \sigma_1 \end{vmatrix}, k \leq n \qquad (5)$$

we obtain

$$\tau_1(1,2,\dots,n) = 1 + 2 + \cdots + n = \frac{1}{2} \cdot n \cdot (n+1)$$

$$\tau_2(1,2,\dots,n) = 1 \cdot 2 + 1 \cdot 3 + \cdots + (n-1) \cdot n$$

$$= \frac{1}{24} \cdot (n-1) \cdot n \cdot (n+1) \cdot (3n+2)$$

$$\tau_3(1,2,\dots,n) = 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 4 + \cdots + (n-2) \cdot (n-1) \cdot n =$$

$$\frac{1}{48} \cdot (n-2) \cdot (n-1) \cdot n^2 \cdot (n+1)^2$$

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$$\tau_n(1,2,\dots,n) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$$

The difficulty of decomposing the right hand sides of the above formulas into factors increases significantly with the growth of *k*.

Proposition 4.

The below recurrence formula, holds

$$\tau_k(1,2,\dots,n) = n \cdot \tau_{k-1}(1,2,\dots,n-1) + \tau_k(1,2,\dots,n-1).$$

Proof.

We have consecutively

$$\tau_k(1,2,\dots,n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} j_1 j_2 \dots j_k =$$

$$n \cdot \sum_{\substack{1 \leq j_1 < j_2 < \cdots < j_{k-1} \leq n-1 \\ j_k = n}} j_1 j_2 \dots j_{k-1} + \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n-1} j_1 j_2 \dots j_k =$$

$$n \cdot \tau_{k-1}(1,2,\dots,n-1) + \tau_k(1,2,\dots,n-1).$$

Finally we obtain the next Stirling triangle for the values of τ_k

1									$n = 0$
1	1								$n = 1$
2	3	1							$n = 2$
6	11	6	1						$n = 3$
24	50	35	10	1					$n = 4$
120	274	225	85	15	1				$n = 5$
720	1764	1624	735	175	21	1			$n = 6$
5040	13068	13132	6769	1960	322	28	1		$n = 7$
40320	109584	118124	67284	22449	4536	546	36	1	$n = 8$
…	…	…	…	…	…	…	…	…	…

Let us go back to formula (5).
Let $\Sigma_0 = 1$,

$$\Sigma_k = \begin{vmatrix} \sigma_1 & 1 & 0 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & 2 & 0 & \cdots & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \sigma_{k-4} & \cdots & k-1 \\ \sigma_k & \sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \cdots & \sigma_1 \end{vmatrix} \quad \text{for } k \geq 1$$

Proposition 5.
The recurrence formula holds

$$\Sigma_k = \sigma_1 \Sigma_{k-1} - (k-1)\sigma_2 \Sigma_{k-2} + (k-1)(k-2)\sigma_3 \Sigma_{k-3} - \ldots \\ +(-1)^{k-1}(k-1)!\,\sigma_{k-1}\Sigma_1 + (-1)^k \sigma_k \Sigma_0 \qquad (6)$$

Proof.
Calculating the above determinant, we obtain

$$\Sigma_k = \sigma_1 \Sigma_{k-1} - (k-1)[\sigma_2 \Sigma_{k-2} - (k-2)[\sigma_3 \Sigma_{k-3} \\ -(k-3)[\ \ldots - 2[\sigma_{k-1}\Sigma_1 - \sigma_k \Sigma_0] \ldots] \\ = \sigma_1 \Sigma_{k-1} - (k-1)\sigma_2 \Sigma_{k-2} + (k-1)(k-2)\sigma_3 \Sigma_{k-3} - \ldots \\ +(-1)^{k-2}(k-1)!\,\sigma_{k-1}\Sigma_1 + (-1)^{k-1}(k-1)!\,\sigma_k \Sigma_0$$

This completes the proof of Proposition 5.

References

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