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VARIATIONAL ANALYSIS OF ONE-DIMENSIONAL NONCONVEX NEUMANN PROBLEM

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Abstract. We consider a variational formulation of a nonconvex one-dimensional Neumann problem. The method of obtaining infimum of a relevant functional is based on a general theorem attributed to Z. Naniewicz, of the minimization of a certain class of nonconvex functionals.

Introduction

In many practical problems, for instance in nonlinear elasticity, we often minimize integral functional I of the form

$$I(u) := \int_{\Omega} f\left(x, u(x), Du(x)\right) dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u: \Omega \to \mathbb{R}^m$ is a function from Sobolev space $W^{1,p}(\Omega), p \geq 1$, (i.e. u and its first distributional derivative belong to $L^p(\Omega)$) and $f: \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}$ is the Carathéodory function. This latter demand means that we want f to be measurable with respect to the first variable and it is continuous with respect to the second and third variables.

The idea of the direct method of minimizing such functionals is to consider a minimizing sequence $(u_n) \subset W^{1,p}(\Omega)$, that is a sequence such that

$$\lim_{n\to\infty} I(u_n) = \inf\{I(u): u \in W^{1,p}(\Omega)\}.$$

What we have to do is to show that (u_n) admits subsequence (u_{n_k}) convergent (in a suitable topology) to some point $u_0 \in W^{1,p}(\Omega)$ and establish the lower semicontinuity of I (with respect to that topology). Then u_0 is a minimum of I because

$$\inf\{I(u): u \in W^{1,p}(\Omega)\} = \lim_{n\to\infty} I(u_n) = \lim_{k\to\infty} I\left(u_{n_k}\right) \geq I(u_0) \geq \inf I.$$

The crucial and most difficult task here is to prove the lower semicontinuity of I.

It is known that if f is the Carathéodory and fulfils suitable growth conditions, then the lower semicontinuity of I is equivalent to the so-called quasiconvexity of f. We refer to [1] for details. However, if f is not quasiconvex, the above procedure is not applicable. One of the most popular methods is to consider relaxed problems: we quasiconvexify the integrand f. The infimum of the relaxed functional remains the same, but some important information concerning the oscillatory nature of the minimizing sequences is lost. This is a serious drawback because from the applicational point of view, the detailed structure of minimizing sequences is often as important as the minimizers themselves. Another problem is that in general it is very difficult to find the relaxation formula for a given nonconvex functional.

In 2001 Z. Naniewicz [2] proved the theorem enabling us to seek minima of functionals with integrands being a minimum of convex functions. He also analyzed the one-dimensional nonconvex Dirichlet problem. In this paper we use his theorem to analyze the one-dimensional nonconvex Neumann problem.

1. Minimization theorem

Let us first introduce some notations. By Ω we will denote the bounded domain in $\mathbb{R}^n$ with the Lipschitz continuous boundary. $H^1(\Omega;\mathbb{R}^m)$ will stand for the Sobolev space of all the functions that are square integrable together with their first partial distributional derivatives. We will also use the abbreviations *a.a.* and *a.e.* instead of *almost all* and *almost everywhere* (with respect to the Lebesgue measure) respectively. For $v \in H^1(\Omega;\mathbb{R}^m)$ we will consider the functional

$$j(v) = \int_{\Omega} \min\{f(x, v(x), Dv(x)), g(x, v(x), Dv(x))\} d\Omega, \tag{1}$$

where $f, g\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ are such that:

(i) $\forall (s,\xi) \in \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ the functions $\Omega \ni x \to f(x,s,\xi)$, $\Omega \ni x \to g(x,s,\xi)$ are measurable;

(ii) for a.a $x \in \Omega$ functions

$$\mathbb{R}^m \times \mathbb{R}^{nm} \ni (s,\xi) \to f(x,s,\xi),\ \mathbb{R}^m \times \mathbb{R}^{nm} \ni (s,\xi) \to g(x,s,\xi)$$

are continuous;

(iii) $a(x) + c(|s|^2 + |\xi|^2) \le f(x,s,\xi) \le A(x) + C(|s|^2 + |\xi|^2)$,
$a(x) + c(|s|^2 + |\xi|^2) \le (x,s,\xi) \le A(x) + C(|s|^2 + |\xi|^2)$,
where $a(\cdot)$ and $A(\cdot)$ are integrable functions in Ω and c, C are positive constants.

Notice also that the left hand side of condition (iii) implies the coercivity of the functional

$$I(u) = \int_\Omega f\big(x, u(x), Du(x)\big)\,dx.$$

We will further write such integrals in the form $\int_\Omega f(u)\,d\Omega$ to simplify the notation.

Remark

In a one-dimensional case, if $u = u(t)$ is the position of the particle moving with the velocity $Du(t) = u'(t)$ and $f(x, s, \xi) = \frac{1}{2}\xi^2$, then functional I describes the kinetic energy of that particle.

Now we state the definition of the quasiconvexity. Let $\omega \subset \mathbb{R}^n$ be a bounded domain, $C_0^1(\omega, \mathbb{R}^m)$ - the space of all vector functions $z = (z_1, \dots, z_n)$ such that for all $i = 1, \dots, n$ function z_i has a compact support in ω and is continuous with its first distributional derivative.

Definition 1

Let $h\colon \mathbb{R}^{nm} \to \mathbb{R}$. We say that h is quasiconvex if for every matrix $\xi \in \mathbb{R}^{nm}$, every ω and every z described above, the following inequality holds:

$$\operatorname{meas}(\omega) \cdot h(\xi) \le \int_\omega h\big(\xi + Dz(x)\big)\,dx,$$

where meas(ω) stands for the Lebesgue measure of the set ω.

Remark

(a) The most typical example of a quasiconvex function is the convex function.

(b) In the scalar case ($n = 1$ or $m = 1$), the notions of convexity and quasiconvexity are equivalent.

(c) The notion of quasiconvexity was introduced by Charles B. Morley in 1952. For it turned out that the assumption of the convexity of the integrand of energy functional I is inconsistent with the principle of the material frame - indifference. See [3] for a review of the topic.

We consider the minimization problem of the form

$$\inf\{J(v)\colon v \in H^1(\Omega; \mathbb{R}^m)\} := \alpha, \qquad (\wp)$$

where J is given by (1):

$$J(v) = \int_\Omega \min\{f(v), g(v)\}\,d\Omega.$$

It is known that if f fulfils conditions $(i), (ii)$ and (iii'): $0 \le f(x, s, \xi) \le A(x) + C(|s|^2 + |\xi|^2)$, then the sequential weak lower semicontinuity of I is equivalent to the quasiconvexity of f with respect to the third variable. However, the integrand of J is not quasiconvex in general, therefore the result is not applicable.

Theorem 2 (Z. Naniewicz 2001)
Suppose that functions f and g are quasiconvex and satisfy (i), (ii) and (iii). Then there exist:
(a) sequence $u_k \in H^1(\Omega;\mathbb{R}^m)$, $u_k \xrightarrow[k\to\infty]{} u$ weakly in $H^1(\Omega;\mathbb{R}^m)$,
(b) sequences $\chi_f^{(k)}: H^1(\Omega;\mathbb{R}^m) \to \{0,1\}$, $\chi_g^{(k)}: H^1(\Omega;\mathbb{R}^m) \to \{0,1\}$ with $\chi_f^{(k)}(u_k)+\chi_g^{(k)}(u_k) \equiv 1$ such that $\chi_f^{(k)} \xrightarrow[k\to\infty]{} \chi_f$, $\chi_g^{(k)} \xrightarrow[k\to\infty]{} \chi_g$, weak* in $L^\infty(\Omega)$, with $\chi_f:\Omega \to [0,1]$, $\chi_g:\Omega \to [0,1]$, and $\chi_f + \chi_g \equiv 1$.
These sequences have the properties:
(c) $\lim_{k\to\infty} \int_\Omega \left[\chi_f^{(k)}(u_k)\mathrm{f}(u_k) + \chi_g^{(k)}(u_k)g(u_k)\right] d\Omega = \alpha;$
(d) $\forall v,w \in H^1(\Omega;\mathbb{R}^m)$ $\int_\Omega \left[\chi_f^{(k)}(u_k)\mathrm{f}(u_k) + \chi_g^{(k)}(u_k)g(u_k)\right] d\Omega \leq$ $\int_\Omega \left[\chi_f^{(k)}(v)\mathrm{f}(w) + \chi_g^{(k)}(v)g(w)\right] d\Omega.$
If additionally
(e) $\limsup \int_\Omega \left[\chi_g^{(k)}(u_k)\big(\mathrm{f}(u_k) - \mathrm{f}(u)\big) + \chi_f^{(k)}(u_k)\big(g(u_k) - g(u)\big)\right] d\Omega \leq 0,$ as $k \to \infty$, then u
is a solution of the primal problem $(\wp)$, i.e. $\int_\Omega \min\big(f(u), g(u)\big) d\Omega = \alpha$ and, moreover, $\lim_{k\to\infty} \int_\Omega \big(\mathrm{f}(u_k) - \mathrm{f}(u)\big)\, d\Omega = 0$ and $\int_\Omega \big(\mathrm{g}(u_k) - \mathrm{g}(u)\big)\, d\Omega = 0.$

The above theorem enables us to introduce term $\mathcal{R}$, called the relaxation term. Namely, we have

$$\int_\Omega \big(\chi_f f(u) + \chi_g g(u)\big) d\Omega - \mathcal{R} = \alpha,$$

and if $\big(\chi_f f(u) + \chi_g g(u)\big) = \min\big(f(u), g(u)\big)$ then $\mathcal{R} = 0$.

In many practical cases it is possible to find the explicit formula for $\mathcal{R} = \mathcal{R}(u, \chi_f, \chi_g)$. This is the subject of the next section.

2. One-dimensional nonconvex Neumann problem

Let $\Omega := I \equiv (0,1)$ and define for $u \in H_0^1(I)$, the functional

$$J(u) := \int_I \min\left\{\tfrac{1}{2}a|u'|^2 + u^2, \tfrac{1}{2}a(|u' - 1|^2 + u^2) + c\right\} dx,$$

Where a, c are real constants, $a > 0$. We consider the following minimization problem:

$$\inf\{J(v): v \in H_0^1(I)\} \equiv \alpha \tag{2}$$

We proceed analogously as in [2]. According to Theorem 2 there sequences (u_k), $\left(\chi_f^{(k)}\right), \left(\chi_g^{(k)}\right)$ such that $\forall w \in H_0^1(I)$ there holds an inequality:

$$\int_I \left\{\chi_f^{(k)} \frac{1}{2} a\left(|u_k'|^2 + u_k^2\right) + \chi_g^{(k)}\left[\frac{1}{2} a\left(|u_k' - 1|^2 + u_k^2\right) + c\right]\right\} dx \le$$

$$\le \int_I \left\{\chi_f^{(k)} \frac{1}{2} a(|w'|^2 + w) + \chi_g^{(k)}\left[\frac{1}{2} a(|w' - 1|^2 + w) + c\right]\right\} dx. \tag{3}$$

This means that u_k is the critical point of the functional

$$J^K(v) := \int_I \left\{\chi_f^{(k)} \frac{1}{2} a(|v'|^2 + v^2) + \chi_g^{(k)}\left[\frac{1}{2} a(|v' - 1|^2 + v^2) + c\right]\right\} dx =$$

$$= \int_I \left\{\chi_f^{(k)} \frac{1}{2} a|v'|^2 + \chi_g^{(k)} \frac{1}{2} a|v' - 1|^2 + \frac{1}{2} a v^2 + \chi_g^{(k)} c\right\}. \tag{4}$$

Therefore $\forall w \in H_0^1(\mathrm{I})$ we have

$$\int_I \left\{\left[\chi_f^{(k)} a u_k' + \chi_g^{(k)} a(u_k' - 1)\right] w' + a u_k w\right\} dx = 0.$$

By the du Bois-Raymond lemma, constant $e_k \in \mathbb{R}$ exists such that

$$\chi_f^{(k)} a u_k' + \chi_g^{(k)} a(u_k' - 1) = \int_0^x a u_k(s) ds + e_k. \tag{5}$$

For $w \in H_0^1(\mathrm{I})$ denote $H(w)(x) := \int_0^x u_k(s) ds$. Equation (4) has now the form

$$\chi_f^{(k)} a u_k' + \chi_g^{(k)} a(u_k' - 1) = H(u_k) + e_k. \tag{6}$$

Multyplying (6) by u_k' and doing some algebraic manipulations we get

$$\chi_f^{(k)} \frac{1}{2} a|u_k'|^2 + \chi_g^{(k)} \frac{1}{2} a|u_k' - 1|^2 + \frac{1}{2} a u_k + \chi_g^{(k)} c =$$

$$\frac{1}{2} H(u_k) u_k' + \frac{1}{2} e_k u_k' - \frac{1}{2} \chi_g^{(k)} a u_k' + \frac{1}{2} \chi_g^{(k)} a + \frac{1}{2} a u_k' + \chi_g^{(k)} c. \tag{7}$$

Observe that the left hand side of (7) is exactly the integrand of J^k given by (4). Now, multiplying (5) by $\chi_g^{(k)}$ and using the fact that $\chi_f^{(k)} \cdot \chi_g^{(k)} = 0$ we get

$$\tfrac{1}{2}\chi_g^{(k)} a u_k' = \tfrac{1}{2}\chi_g^{(k)} H(u_k) + \tfrac{1}{2}\chi_g^{(k)} e_k + \tfrac{1}{2}\chi_g^{(k)} a,$$

which inserted into (7) leads to

$$J^{(k)}(u_k) = \int_I \left(\chi_f^{(k)} \tfrac{1}{2} a |u_k'|^2 + \chi_g^{(k)} \tfrac{1}{2} a |u_k' - 1|^2 + \tfrac{1}{2} a u_k + \chi_g^{(k)} c\right) dx =$$

$$= \int_I \left(\tfrac{1}{2} H(u_k) u_k' - \tfrac{1}{2}\chi_g^{(k)} H(u_k) - \tfrac{1}{2}\chi_g^{(k)} e_k + \chi_g^{(k)} c + \tfrac{1}{2} a u_k^2 + \tfrac{1}{2} e_k u_k'\right) dx \quad (8)$$

Sequence (e_k) as bounded in $\mathbb{R}$ has a subsequence convergent to some $e \in \mathbb{R}$. By the Sobolev imbedding theorem, sequence (u_k) is convergent to some u in $C(I)$. This means that we can pass to the limit as $k \to \infty$ in (8) and get

$$J(u) = \alpha = \int_I \left(\tfrac{1}{2} H(u) u' - \tfrac{1}{2}\chi_g H(u) - \tfrac{1}{2}\chi_g e + \chi_g c + \tfrac{1}{2} a u^2 + \tfrac{1}{2} e u'\right) dx \quad (9)$$

Again from (6) we get

$$\chi_f^{(k)} a u_k' = H(u_k)\chi_f^{(k)} + \chi_f^{(k)} e_k$$

and

$$\chi_g^{(k)} a u_k' = H(u_k)\chi_g^{(k)} + \chi_g^{(k)} e_k + \chi_g^{(k)} e_k.$$

Adding the above inequalities yields

$$u_k' = \tfrac{1}{a} H(u_k) + \tfrac{1}{a} e_k + \chi_g^{(k)}.$$

As each term in this equality is weakly convergent in $L^2(I)$, we can pass to ∞ with k:

$$u' = \tfrac{1}{a} H(u) + \tfrac{1}{a} e + \chi_g. \quad (10)$$

With the help of (10) we can then write

$$\tfrac{1}{2} a \chi_f |u'|^2 = \tfrac{1}{2}\chi_f H(u) u' + \tfrac{1}{2}\chi_f e u' + \tfrac{1}{2}\chi_f a u^2 + \tfrac{1}{2}\chi_f \chi_g a u'$$

and

$$\tfrac{1}{2}a\chi_g|u'-1|^2 = \tfrac{1}{2}\chi_g H(u)u' - \tfrac{1}{2}\chi_g H(u) + \tfrac{1}{2}\chi_g eu' - \tfrac{1}{2}\chi_g e -$$
$$-\tfrac{1}{2}au'\chi_f\chi_g + \tfrac{1}{2}a\chi_f\chi_g,$$

and further

$$\tfrac{1}{2}a\chi_f|u'|^2 + \tfrac{1}{2}a\chi_g|u'-1|^2 + \tfrac{1}{2}au^2 + \chi_g c =$$

$$= \tfrac{1}{2}H(u)u' - \tfrac{1}{2}\chi_g H(u) + \tfrac{1}{2}eu' - \tfrac{1}{2}\chi_g e + \tfrac{1}{2}au^2 + \chi_g c + \tfrac{1}{2}\chi_g\chi_f a.$$

Observe that the right hand side of the above equation is in fact the integrand in (9) plus $\chi_g c$ and $\frac{1}{2}\chi_g\chi_f a$.

We can now write the expression for the infimum of J:

$$\alpha = \int_I \left[\tfrac{1}{2}H(u)u' - \tfrac{1}{2}\chi_g H(u) + \tfrac{1}{2}eu' - \tfrac{1}{2}\chi_g e + \tfrac{1}{2}au^2 + \chi_g c\right]dx =$$

$$= \int_I \left[\tfrac{1}{2}a\chi_f|u'|^2 + \tfrac{1}{2}a\chi_g|u'-1|^2 + \tfrac{1}{2}au^2 + \chi_g c\right]dx - \mathcal{R}(\chi_f,\chi_g), \quad (11)$$

where

$$\mathcal{R}(\chi_f,\chi_g) := \int_I \tfrac{1}{2}\chi_g\chi_f a\,dx = \tfrac{1}{2}\chi_g\chi_f a$$

is the relaxation term. Of course $\mathcal{R}$ takes its maximum value $\frac{1}{8}a$ if $\chi_f = \chi_g = \frac{1}{2}$. If $\chi_f = 0$ or $\chi_g = 0$, then the relaxation term vanishes and we have the solution of the classical one-dimensional Neumann problem. In paticular, this is the case if sequence $\chi_f^{(k)}$ converges strongly to 1 (or 0) i.e. if it is constant for all k larger than some k_0.

References

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