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TIME-DEPENDENT STATE PROBABILITIES OF QUEUEING NETWORK WITH UNRELIABLE SYSTEMS IN TRANSIENT REGIME

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Abstract. The method of a multidimensional generating function for finding time-dependent state probabilities of a open queuing network with unreliable systems is studied. Assuming that the network functions in the condition of heavy loading, the flow parameters of messages, servicing, duration of serviceable working and duration of restoration of channels are time-dependent. Such networks can serve as models of the functioning of the Local Area Network (LAN). Expressions for network state probabilities at any moment of time are obtained. An example for finding network state probabilities with a central system is considered.

Introduction

Queueing networks (QN) can be used as stochastic models of various computer systems and networks, various objects in economy, production, insurance, medicine and other fields. QN with unreliable systems are described in [1], where formulas for their stationary state probability are resulted. In this article by using generating functions expressions for time-dependent state probabilities, the average number of messages and serviceable channels are obtained.

Let us examine an open exponential QN with one type of messages that consist of n queueing systems (QS) $S_1, S_2, \ldots, S_n$. The Poisson flow of one type of messages with arrival rate $\lambda(t)$ comes into the network from outside. Let system S_i consist of m_i identical service channels, the service time in each of which has an exponential distribution with parameter $\mu_i(t)$, $i = \overline{1,n}$.

Let us suppose that the service channels of system S_0 are absolutely reliable. At the other QS, $S_1, S_2, \ldots, S_n$ service channels are exposed to random failure and serviceable work time of each channel of the system, S_i has an exponential distribution with parameter $\beta_i(t)$, $i = \overline{1,n}$. After failure, the service channel immediately starts to be restored and the restoration time also has an exponential distribution with parameter $\gamma_i(t)$, $i=\overline{1,n}$. Let us consider that the service times of messages, durations of serviceable work of channels and restoration time of service channels are independent random variables. The state of such a network could be described via vector

$$Z(t)=(z,t)=(d,k,t)=(d_1,d_2,\ldots,d_n,k_1,k_2,\ldots,k_n,t),$$

where: d_i - number of serviceable channels in system S_i, $0\le d_i\le m_i$, k_i - the number of messages in system S_i at moment t, $t\in[0,+\infty)$, $i=\overline{1,n}$. Let p_{0j} - the probability of a message entering from outside to system S_j, $\sum_{j=1}^{n}p_{0j}=1$; the probability of message transition from system S_i to system S_j, $\sum_{j=0}^{n}p_{ij}=1$, $i=\overline{1,n}$, $u(x)=\begin{cases}1, & x>0\\ 0, & x\le 0\end{cases}$ - the Heaviside function. Matrix $P=\|p_{ij}\|_{(n+1)\times(n+1)}$ is the matrix of passage probabilities of irreducible Markovian chains. The service rate of a message occurs according to discipline FIFO.

Thus, the case when the flow parameters of messages, servicing, duration of serviceable working and duration of the restoration of channels are time-dependent is considered. In time interval $[t,t+\Delta t)$ a message will arrival in QN with probability $\lambda(t)\Delta t+o(\Delta t)$; if at moment t a message is detected for service in the channel of i - s QS then in time interval $[t,t+\Delta t)$ it will be serviced with probability $\mu_i(t)\Delta t+o(\Delta t)$, $i=\overline{1,n}$; in time interval $[t,t+\Delta t)$, the channels of i - s QS with probability $\beta_i(t)\Delta t+o(\Delta t)$, breakage occurs or starts to be restored with probability $\gamma_i(t)\Delta t+o(\Delta t)$, $i=\overline{1,n}$.

Lemma. The state probabilities of the network under review satisfy the system of difference-differential equations (DDE):

$$\frac{dP(d,k,t)}{dt}=-\left[\lambda(t)+\sum_{i=1}^{n}\left[\mu_i(t)\min(d_i,k_i)+\beta_i(t)d_i+\gamma_i(t)(m_i-d_i)\right]\right]P(d,k,t)+$$

$$+\lambda(t)\sum_{i=1}^{n}p_{0i}u(k_i)P(d,k-I_i,t)+$$

$$+\sum_{i=1}^{n}\mu_i(t)\min(d_i,k_i+1)p_{i0}P(d,k+I_i,t)+$$

$$+\sum_{i,j=1}^{n}\mu_i(t)\min(d_i,k_i+1)p_{ij}u(k_j)P(d,k+I_i-I_j,t)+$$

$$+\sum_{i=1}^{n}\gamma_i(t)(m_i-d_i+1)u(d_i)P(d-I_i,k,t)+\sum_{i=1}^{n}\beta_i(t)(d_i+1)P(d+I_i,k,t) \quad (1)$$

Finding state probabilities with help of generating function method

Let denote $\Psi_{2n}(z,t)$, where $z=(z_1,z_2,\ldots,z_n,z_{n+1},\ldots,z_{2n})$, $2n$ - dimensional generating function

$$\Psi_{2n}(z,t)=\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}P(d_1,\ldots,d_n,k_1,\ldots,k_n,t)z_1^{d_1}\ldots z_n^{d_n}z_{n+1}^{k_1}\ldots z_{2n}^{k_n}=$$

$$=\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}P(d,k,t)\prod_{i=1}^{n}z_i^{d_i}z_{n+i}^{k_i}\,,\ |z|<1\,. \quad (2)$$

Let us suppose that all network systems function in heavy loading [2], i.e. $k_i(t)>d_i(t)$ $\forall t>0$, $i=\overline{1,n}$. Then system (1) will take the form

$$\frac{dP(d,k,t)}{dt}=-\left[\lambda(t)+\sum_{i=1}^{n}\left[\left(\mu_i(t)+\beta_i(t)-\gamma_i(t)\right)d_i+\gamma_i(t)m_i\right]\right]P(d,k,t)+$$

$$+\lambda(t)\sum_{i=1}^{n}p_{0i}P(d,k-I_i,t)+\sum_{i=1}^{n}\mu_i(t)d_ip_{i0}P(d,k+I_i,t)+$$

$$+\sum_{i,j=1}^{n}\mu_i(t)d_ip_{ij}P(d,k+I_i-I_j,t)+\sum_{i=1}^{n}\gamma_i(t)(m_i-d_i+1)u(d_i)P(d-I_i,k,t)+$$

$$+\sum_{i=1}^{n}\beta_i(t)(d_i+1)P(d+I_i,k,t)\,. \quad (3)$$

Note that the equation number of system (2) is countable when the network is open and certain when it is closed.

Theorem 1. Generating function $\Psi_{2n}(z,t)$ satisfies the partial differential equation (PDE):

$$\frac{\partial\Psi_{2n}(z,t)}{\partial t}=-\left[\lambda(t)\left(1-\sum_{i=1}^{n}p_{0i}z_{n+i}\right)+\sum_{i=1}^{n}\gamma_i(t)m_i(1-z_i)\right]\Psi_{2n}(z,t)-$$

$$-\sum_{i=1}^{n}\left[\left(\mu_i(t)+\beta_i(t)\right)z_i-\mu_i(t)\frac{p_{i0}}{z_{n+i}}-\frac{\beta_i(t)}{z_i}\right]\frac{\partial\Psi_{2n}(z,t)}{\partial z_i}+$$

$$+\sum_{i,j=1}^{n}\mu_i(t)\,p_{ij}\frac{z_{n+j}}{z_{n+i}}\frac{\partial\Psi_{2n}(z,t)}{\partial z_i}\,. \quad (4)$$

Proof. Multiply (3) by $\prod_{l=1}^{n} z_l^{d_l} z_l^{k_l}$ and the sum by all possible values d_l from 0 to m_l and k_l from 1 to $+\infty$, $l=\overline{1,n}$. Here the summation on all k_l is accepted from 1, since all terms in (2), for which in the state of network $Z(t)$ there are components $k_l=0$, because the network systems function in heavy loading, for example, $P(d,k_1,\ldots,k_{l-1},0,k_{l+1},\ldots,k_n,t)=0$, $l=\overline{2,n}$. Then

$$\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}\frac{dP(d,k,t)}{dt}\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l} =$$

$$=-\left[\lambda(t)+\sum_{i=1}^{n}\Big[\big(\mu_i(t)+\beta_i(t)-\gamma_i(t)\big)d_i+\gamma_i(t)m_i\Big]\right]\times$$

$$\times\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k,t)\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l}+$$

$$+\lambda(t)\sum_{i=1}^{n}p_{0i}u(k_i)\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k-I_i,t)\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l}+$$

$$+\sum_{i=1}^{n}\mu_i(t)d_i p_{i0}\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k+I_i,t)\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l}+$$

$$+\sum_{i,j=1}^{n}\mu_i(t)d_i p_{ij}u(k_j)\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k+I_i-I_j,t)\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l}+$$

$$+\sum_{i=1}^{n}\gamma_i(t)(m_i-d_i+1)u(d_i)\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d-I_i,k,t)\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l}+$$

$$+\sum_{i=1}^{n}\beta_i(t)(d_i+1)u(m_i-d_i)\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d+I_i,k,t)\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l} \tag{5}$$

Let us examine some sums that enter the right part of relation (5). Let

$$\sum\nolimits_1(z,t)=-\left[\lambda(t)+\sum_{i=1}^{n}\Big[\big(\mu_i(t)+\beta_i(t)-\gamma_i(t)\big)d_i+\gamma_i(t)m_i\Big]\right]\times$$

$$\times\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k,t)\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l}.$$

Then

$$\sum\nolimits_1(z,t)=-\left[\lambda(t)+\sum_{i=1}^{n}\gamma_i(t)m_i\right]\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k,t)\prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l}-$$

$$-\sum_{i=1}^{n}\left(\mu_i(t)+\beta_i(t)-\gamma_i(t)\right)d_i\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\mathrm{K}\sum_{k_n=1}^{\infty}P(d,k,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}=$$

$$=-\left[\lambda(t)+\sum_{i=1}^{n}\gamma_i(t)m_i\right]\Psi_{2n}(z,t)-\sum_{i=1}^{n}\left(\mu_i(t)+\beta_i(t)-\gamma_i(t)\right)z_i\frac{\partial\Psi_{2n}(z,t)}{\partial z_i}$$

For sum $\sum_2(z,t)=\lambda(t)\sum_{i=1}^{n}p_{0i}u(k_i)\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\mathrm{K}\sum_{k_n=1}^{\infty}P(d,k-I_i,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}$

we have

$$\sum_2(z,t)=\lambda(t)\sum_{i=1}^{n}p_{0i}z_{n+i}\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{\substack{k_j=1\\ j=\overline{1,n},\,j\neq i}}^{\infty}\sum_{k_i=1}^{\infty}P(d,k-I_i,t)\frac{\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}}{z_{n+i}}=$$

$$=\lambda(t)\sum_{i=1}^{n}p_{0i}z_{n+i}\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{\substack{k_j=1\\ j=\overline{1,n}}}^{\infty}P(d,k,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}=\lambda(t)\sum_{i=1}^{n}p_{0i}z_{n+i}\Psi_{2n}(z,t)$$

For sum $\sum_3(z,t)=\sum_{i=1}^{n}\mu_i(t)d_ip_{i0}\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\mathrm{K}\sum_{k_n=1}^{\infty}P(d,k+I_i,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}$

looks like:

$$\sum_3(z,t)=\sum_{i=1}^{n}\mu_i(t)d_i\frac{p_{i0}}{z_{n+i}}\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\mathrm{K}\sum_{k_n=1}^{\infty}P(d,k+I_i,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}z_{n+i}=$$

$$=\sum_{i=1}^{n}\mu_i(t)\frac{p_{i0}}{z_{n+i}}\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\mathrm{K}\sum_{k_n=1}^{\infty}d_iP(d,k,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}-$$

$$-\sum_{i=1}^{n}\mu_i(t)\frac{p_{i0}}{z_{n+i}}\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{\substack{k_j=1\\ j=\overline{1,n},\,j\neq i}}^{\infty}d_iP(d,k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n,t)\frac{\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}}{z_{n+i}^{k_i}}=$$

$$=\sum_{i=1}^{n}\mu_i(t)\frac{p_{i0}}{z_{n+i}}\frac{\partial\Psi_{2n}(z,t)}{\partial z_i}-$$

$$-\sum_{i=1}^{n}\mu_i(t)\frac{p_{i0}}{z_{n+i}}\sum_{d_1=0}^{m_1}\mathrm{K}\sum_{d_n=0}^{m_n}\sum_{\substack{k_j=1\\ j=\overline{1,n},\,j\neq i}}^{\infty}d_iP(d,k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n,t)\frac{\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}}{z_{n+i}^{k_i}}=$$

$$=\sum_{i=1}^{n}\mu_i(t)\frac{p_{i0}}{z_{n+i}}\frac{\partial\Psi_{2n}(z,t)}{\partial z_i}.$$

For sum

$$\sum\nolimits_4(z,t)=\sum_{i,j=1}^{n}\mu_i(t)d_i p_{ij}u(k_j)\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k+I_i-I_j,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}=$$

$$=\sum_{i,j=1}^{n}\mu_i(t)d_i p_{ij}\frac{z_{n+j}}{z_{n+i}}\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k+I_i-I_j,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}\cdot\frac{z_{n+i}}{z_{n+j}}=$$

$$=\sum_{i,j=1}^{n}\mu_i(t)p_{ij}\frac{z_{n+j}}{z_{n+i}}\frac{\partial\Psi_{2n}(z,t)}{\partial z_i}-\sum_{i,j=1}^{n}\mu_i(t)p_{ij}\frac{z_{n+j}}{z_{n+i}}\times$$

$$\times\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{\substack{k_j=1\\ j=\overline{1,n},\,j\neq i}}^{\infty}d_iP(d,k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}\cdot\frac{z_{n+i}}{z_{n+j}}=$$

$$=\sum_{i,j=1}^{n}\mu_i(t)p_{ij}\frac{z_{n+j}}{z_{n+i}}\frac{\partial\Psi_{2n}(z,t)}{\partial z_i}.$$

For sum

$$\sum\nolimits_5(z,t)=\sum_{i=1}^{n}\gamma_i(t)(m_i-d_i+1)u(d_i)\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d-I_i,k,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}=$$

$$=\left[\sum_{i=1}^{n}\gamma_i(t)m_i-\sum_{i=1}^{n}\gamma_i(t)u(d_i)(d_i-1)\right]\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d-I_i,k,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}=$$

$$=\left[\sum_{i=1}^{n}\gamma_i(t)\big(m_i-u(d_i)(d_i-1)\big)\right]z_i\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d-I_i,k,t)\frac{\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}}{z_i}=$$

$$=\left[\sum_{i=1}^{n}\gamma_i(t)(m_i-d_i)\right]z_i\sum_{\substack{d_j=0\\ j=\overline{1,n}\ j\neq i}}^{m_j}\sum_{d_i=0}^{m_i-1}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}=$$

$$=\left[\sum_{i=1}^{n}\gamma_i(t)(m_i-d_i)\right]z_i\sum_{d_1=0}^{m_1}\ldots\sum_{d_n=0}^{m_n}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d,k,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}-$$

$$-\left[\sum_{i=1}^{n}\gamma_i(t)(m_i-m_i)\right]z_i\sum_{\substack{d_j=0\\ j=\overline{1,n}\ j\neq i}}^{m_j}\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}P(d_1,\ldots,d_{i-1},m_i,d_{i+i},\ldots,d_n,k,t)\prod_{l=1}^{n}z_l^{d_l}z_{n+l}^{k_l}=$$

$$=\sum_{i=1}^{n}\gamma_i(t)m_iz_i\Psi_{2n}(z,t)-\sum_{i=1}^{n}\gamma_i(t)z_i\frac{\partial\Psi_{2n}(z,t)}{\partial z_i}.$$

And finally for the last sum we will have

$$\sum\nolimits_6 (z,t) = \sum_{i=1}^{n} \beta_i(t)(d_i+1) \sum_{d_1=0}^{m_1} \ldots \sum_{d_n=0}^{m_n} \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} P(d+I_i,k,t) \prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l} =$$

$$= \sum_{i=1}^{n} \frac{\beta_i(t)}{z_i} \sum_{d_1=0}^{m_1} \ldots \sum_{d_n=0}^{m_n} \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} (d_i+1) P(d+I_i,k,t) \prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l} z_i =$$

$$= \sum_{i=1}^{n} \frac{\beta_i(t)}{z_i} \sum_{\substack{d_j=0 \\ j=\overline{1,n},\, j\neq i}}^{m_j} \sum_{d_i=1}^{m_i} \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} d_i P(d,k,t) \prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l} z_i =$$

$$= \sum_{i=1}^{n} \frac{\beta_i(t)}{z_i} \sum_{d_1=0}^{m_1} \ldots \sum_{d_n=0}^{m_n} \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} d_i P(d,k,t) \prod_{l=1}^{n} z_l^{d_l} z_{n+l}^{k_l} = \sum_{i=1}^{n} \frac{\beta_i(t)}{z_i} \frac{\partial \Psi_{2n}(z,t)}{\partial z_i}.$$

Thus, considering the kind of generating function (2), we will have the first-order PDE.

Let us examine the case when

$$m_i = 1,\ k_i(t) > 0 \ \ \forall t,\ i = \overline{1,n}. \qquad (6)$$

The number of serviceable channels in system S_i can be equal to 0 or 1. If the state of network (d,k,t) is $(d_1,\ldots,d_{i-1},0,d_{i+1},\ldots,d_n,k,t)$, then the right system of equations

$$\frac{dP(d,k,t)}{dt} = -\left[\lambda(t) + \sum_{i=1}^{n}\left[\mu_i(t)+\beta_i(t)+\gamma_i(t)\right]\right] P(d,k,t) +$$

$$+\lambda(t)\sum_{i=1}^{n} p_{0i} P(d,k-I_i,t) + \sum_{i=1}^{n} \mu_i(t) p_{i0} P(d,k+I_i,t) +$$

$$+\sum_{i,j=1}^{n} \mu_i(t) p_{ij} P(d,k+I_i-I_j,t) + \sum_{i=1}^{n} \beta_i(t) P(d_1,\ldots,d_{i-1},1,d_{i+i},\ldots,d_n,k,t),$$

and if $(d_1,\ldots,d_{i-1},1,d_{i+1},\ldots,d_n,k,t)$, then

$$\frac{dP(d,k,t)}{dt} = -\left[\lambda(t) + \sum_{i=1}^{n}\left[\mu_i(t)+\beta_i(t)+\gamma_i(t)\right]\right] P(d,k,t) +$$

$$+\lambda(t)\sum_{i=1}^{n} p_{0i} P(d,k-I_i,t) + \sum_{i=1}^{n} \mu_i(t) p_{i0} P(d,k+I_i,t) +$$

$$+\sum_{i,j=1}^{n} \mu_i(t) p_{ij} P(d,k+I_i-I_j,t) + \sum_{i=1}^{n} \gamma_i(t) P(d_1,\ldots,d_{i-1},0,d_{i+i},\ldots,d_n,k,t).$$

They can be combined into one system

$$\frac{dP(d,k,t)}{dt}=-\left[\lambda(t)+\sum_{i=1}^{n}\left[\mu_i(t)+\beta_i(t)+\gamma_i(t)\right]\right]P(d,k,t)+$$

$$+\lambda(t)\sum_{i=1}^{n}p_{0i}P(d,k-I_i,t)+\sum_{i=1}^{n}\mu_i(t)p_{i0}P(d,k+I_i,t)+$$

$$+\sum_{i,j=1}^{n}\mu_i(t)\,p_{ij}P\left(d,k+I_i-I_j,t\right)+\sum_{i=1}^{n}\gamma_i(t)u(d_i)P(d_1,\dots,d_{i-1},0,d_{i+i},\dots,d_n,k,t)+$$

$$+\sum_{i=1}^{n}\beta_i(t)\left(1-u(d_i)\right)P(d_1,\dots,d_{i-1},1,d_{i+i},\dots,d_n,k,t). \qquad (7)$$

Let

$$\Lambda(t)=\int\lambda(t)dt\,,\ \mathrm{M}_i(t)=\int\mu_i(t)dt\,,$$

$$\mathrm{B}_i(t)=\int\beta_i(t)dt\,,\ \mathrm{G}_i(t)=\int\gamma_i(t)dt\,. \qquad (8)$$

Theorem 2. If QN at the initial moment is in state $(\alpha_1,\alpha_2,\dots,\alpha_{2n},0)$, $\alpha_i\ge 0$, $\alpha_{n+i}>0$, $i=\overline{1,n}$, then generating function (2) can be written as

$$\Psi_{2n}(z,t)=a_0(t)\exp\left\{\left(\Lambda(t)-\Lambda(0)\right)\sum_{i=1}^{n}p_{0i}z_{n+i}\right\}\exp\left\{\sum_{i=1}^{n}\left(\mathrm{M}_i(t)-\mathrm{M}_i(0)\right)\frac{p_{i0}}{z_{n+i}}\right\}\times$$

$$\times\exp\left\{\sum_{i,j=1}^{n}\left(\mathrm{M}_i(t)-\mathrm{M}_i(0)\right)p_{ij}\frac{z_{n+j}}{z_{n+i}}\right\}\exp\left\{\sum_{i=1}^{n}\left(\mathrm{G}_i(t)-\mathrm{G}_i(0)\right)\left(1-u(d_i)\right)z_i\right\}\times$$

$$\times\exp\left\{\sum_{i=1}^{n}\left(\mathrm{B}_i(t)-\mathrm{B}_i(0)\right)u(d_i)\frac{1}{z_i}\right\}\prod_{l=1}^{2n}z_l^{\alpha_l} \qquad (9)$$

where

$$a_0(t)=\exp\Big\{-\left(\Lambda(t)-\Lambda(0)\right)-$$

$$-\sum_{i=1}^{n}\left[\left(\mathrm{M}_i(t)-\mathrm{M}_i(0)\right)+\left(\mathrm{B}_i(t)-\mathrm{B}_i(0)\right)+\left(\mathrm{G}_i(t)-\mathrm{G}_i(0)\right)\right]\Big\} \qquad (10)$$

Let us set out the exponents in (9) in a Maclaurin series to transform it to a form which is suitable for finding the state probabilities of the network. Then the following statement is true.

Theorem 3. The expression for generating function (9) has the form

$$\Psi_{2n}(z,t)=a_0(t)\sum_{g_1=0}^{\infty}\ldots\sum_{g_n=0}^{\infty}\sum_{q_1=0}^{\infty}\ldots\sum_{q_n=0}^{\infty}\sum_{l_1=0}^{\infty}\ldots\sum_{l_n=0}^{\infty}\sum_{r_1=0}^{\infty}\ldots\sum_{r_n=0}^{\infty}\sum_{h_1=0}^{\infty}\ldots\sum_{h_n=0}^{\infty}\left(\Lambda(t)-\Lambda(0)\right)^{\sum_{i=1}^{n}l_i}\times$$

$$\times\prod_{i=1}^{n}\left[\frac{p_{0i}^{l_i}p_{i0}^{r_i}\left(\prod_{j=1}^{n}p_{ij}\right)^{h_i}}{l_i!r_i!h_i!q_i!g_i!}\left[\mathrm{M}_i(t)-\mathrm{M}_i(0)\right]^{r_i+h_i}\left[\left(\mathrm{G}_i(t)-\mathrm{G}_i(0)\right)\left(1-u(d_i)\right)\right]^{q_i}\times\right.$$

$$\left.\times\left[\left(\mathrm{B}_i(t)-\mathrm{B}_i(0)\right)u(d_i)\right]^{g_i}z_i^{\alpha_i+q_i-g_i}z_{n+i}^{\alpha_{n+i}+l_i-r_i-h_i+H}\right], \qquad (11)$$

where $H=\sum_{i=1}^{n}h_i$.

Example 1. Let us examine the LAN model which is represented in Figure 1. Systems $S_1,S_2,\ldots,S_{n-1}$ correspond to the data terminal equipment (peripheral computers), system S_n – local server. Remember that LAN often functions in a condition of heavy loading [2]. An inquiry (packets, messages) can enter the server not only from terminals but from outside the medium via a base station.

Expression (11) in this case takes the form

$$\Psi_{2n}(z,t)=a_0(t)\sum_{g_1=0}^{\infty}\ldots\sum_{g_n=0}^{\infty}\sum_{q_1=0}^{\infty}\ldots\sum_{q_n=0}^{\infty}\sum_{l_1=0}^{\infty}\ldots\sum_{l_n=0}^{\infty}\sum_{r_1=0}^{\infty}\ldots\sum_{r_n=0}^{\infty}\sum_{h_1=0}^{\infty}\ldots\sum_{h_n=0}^{\infty}\left(\Lambda(t)-\Lambda(0)\right)^{\sum_{i=1}^{n}l_i}\times$$

$$\times\prod_{j=1}^{n-1}p_{jn}^{h_j}p_{nj}^{h_n}\prod_{i=1}^{n}\left[\frac{p_{0i}^{l_i}p_{i0}^{r_i}}{l_i!r_i!h_i!q_i!g_i!}\left[\mathrm{M}_i(t)-\mathrm{M}_i(0)\right]^{r_i+h_i}\left[\left(\mathrm{G}_i(t)-\mathrm{G}_i(0)\right)\left(1-u(d_i)\right)\right]^{q_i}\times\right.$$

$$\left.\times\left[\left(\mathrm{B}_i(t)-\mathrm{B}_i(0)\right)u(d_i)\right]^{g_i}z_i^{\alpha_i+q_i-g_i}z_{n+i}^{\alpha_{n+i}+l_i-r_i-h_i+H}\right] \qquad (12)$$

Let, for example,

$$\lambda(t)=\lambda\cos(at+\omega)+b\,,\ \mu_i(t)=\mu_i\sin(a_it+\omega_i)+c_i\,,$$

$$\beta_i(t)=\beta_i\sin(\theta_it+\nu_i)+\rho_i\,,\ \gamma_i(t)=\gamma_i\cos(\eta_it+\delta_i)+e_i\,,\ i=\overline{1,n}$$

then

$$\Lambda(t)=\frac{\lambda\sin(at+\omega)}{a}+bt\ ,\ \Lambda(0)=\frac{\lambda\sin\omega}{a}$$

$$\mathrm{M}_i(t)=-\mu_i\frac{\cos(a_it+\omega_i)}{a_i}+c_it\ ,\ \mathrm{M}_i(0)=-\mu_i\frac{\cos\omega_i}{a_i}$$

$$\mathrm{B}_i(t)=-\beta_i\frac{\cos(\theta_it+\nu_i)}{\theta_i}+\rho_it\ ,\ \mathrm{B}_i(0)=-\beta_i\frac{\cos\nu_i}{\theta_i}$$

$$\mathrm{G}_i(t)=-\gamma_i\frac{\cos(\eta_it+\delta_i)}{\eta_i}+e_it\ ,\ \mathrm{G}_i(0)=-\gamma_i\frac{\cos\delta_i}{\eta_i}\ ,\ i=\overline{1,n}$$

$$a_0(t)=\exp\left\{-\left(bt+\frac{\lambda\sin(at+\omega)}{a}-\frac{\lambda\sin\omega}{a}\right)-\right.$$

$$-\sum_{i=1}^{n}\left[\left(c_it-\mu_i\frac{\cos(a_it+\omega_i)}{a_i}+\mu_i\frac{\cos\omega_i}{a_i}\right)+\right.$$

$$\left.\left.+\left(e_it-\gamma_i\frac{\cos(\eta_it+\delta_i)}{\eta_i}+\gamma_i\frac{\cos\delta_i}{\eta_i}\right)+\left(\rho_it-\beta_i\frac{\cos(\theta_it+\nu_i)}{\theta_i}+\beta_i\frac{\cos\nu_i}{\theta_i}\right)\right]\right\}.$$

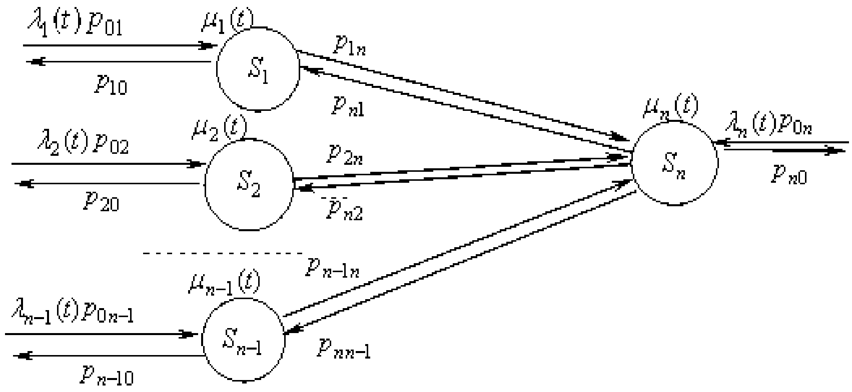

Fig. 1. Model of LAN

From (12) we have

$$\Psi_{2n}(z,t)=a_0(t)\times$$

$$\times\sum_{g_1=0}^{\infty}\text{K}\sum_{g_n=0}^{\infty}\sum_{q_1=0}^{\infty}\text{K}\sum_{q_n=0}^{\infty}\sum_{l_1=0}^{\infty}\text{K}\sum_{l_n=0}^{\infty}\sum_{r_1=0}^{\infty}\text{K}\sum_{r_n=0}^{\infty}\sum_{h_1=0}^{\infty}\text{K}\sum_{h_n=0}^{\infty}\left(bt+\frac{\lambda\sin(at+\omega)}{a}-\frac{\lambda\sin\omega}{a}\right)^{\sum_{i=1}^{n}l_i}\times$$

$$\times\prod_{j=1}^{n-1} p_{jn}^{h_j} p_{nj}^{h_n} \prod_{i=1}^{n}\left[\frac{p_{0i}^{l_i} p_{i0}^{r_i}}{l_i!r_i!h_i!q_i!g_i!}\left[c_i t-\mu_i\frac{\cos\left(a_i t+c_i\right)}{a_i}+\mu_i\frac{\cos\omega_i}{a_i}\right]^{r_i+h_i}\times\right.$$

$$\times\left[\left(e_i t-\gamma_i\frac{\cos\left(\eta_i t+\delta_i\right)}{\eta_i}+\gamma_i\frac{\cos\delta_i}{\eta_i}\right)\left(1-u\left(d_i\right)\right)\right]^{q_i}\times$$

$$\left.\times\left[\left(\rho_i t-\beta_i\frac{\cos\left(\theta_i t+\nu_i\right)}{\theta_i}+\beta_i\frac{\cos\nu_i}{\theta_i}\right)u\left(d_i\right)\right]^{g_i} z_i^{\alpha_i+q_i-g_i} z_{n+i}^{\alpha_{n+i}+l_i-r_i-h_i+H}\right]$$

State probability $P\left(d_1\ldots,d_n,k_1,\ldots,k_n,t\right)$ is a coefficient at $z_1^{d_1}\cdot\ldots\cdot z_n^{d_n} z_{n+1}^{k_1}\cdot\ldots\cdot z_{2n}^{k_n}$ in the decomposition of function $\Psi_{2n}\left(z,t\right)$ in multiple series (12), if QN at the initial moment is in state $\left(\alpha_1,\alpha_2,\ldots,\alpha_{2n},0\right)$. Hence the degrees at z_i should satisfy relation $\alpha_i+q_i-g_i=d_i$, and at z_{n+i}, relation $\alpha_{n+i}+l_i-r_i-h_i+H=k_i$, $i=\overline{1,n}$. Then

$$\alpha_{n+i}+l_i-r_i+\sum_{\substack{j=1\\ j\neq i}}^{n} h_j=k_i,\ i=\overline{1,n}$$

$$r_i=\alpha_{n+i}+l_i+\sum_{\substack{j=1\\ j\neq i}}^{n} h_j-k_i,\ i=\overline{1,n}$$

$$r_i+h_i=\alpha_{n+i}+l_i+\sum_{i=1}^{n} h_i-k_i,\ i=\overline{1,n}$$

and state probability $\left(d_1\ldots,d_n,k_1,\ldots,k_n,t\right)$ can be calculated:

$$P\left(d_1\ldots,d_n,k_1,\ldots,k_n,t\right)=a_0\left(t\right)\times$$

$$\times\sum_{g_1=0}^{\infty}\ldots\sum_{g_n=0}^{\infty}\sum_{q_1=0}^{\infty}\ldots\sum_{q_n=0}^{\infty}\sum_{l_1=0}^{\infty}\ldots\sum_{l_n=0}^{\infty}\sum_{h_1=0}^{\infty}\ldots\sum_{h_n=0}^{\infty}\left(bt+\frac{\lambda\sin\left(at+\omega\right)}{a}-\frac{\lambda\sin\omega}{a}\right)^{\sum_{i=1}^{n} l_i}\prod_{j=1}^{n-1} p_{jn}^{h_j} p_{nj}^{h_n}\times$$

$$\times\prod_{i=1}^{n}\left[\frac{p_{0i}^{l_i} p_{i0}^{\alpha_{n+i}+l_i+\sum_{\substack{j=1\\ j\neq i}}^{n} h_j-k_i}}{l_i!h_i!q_i!g_i!\left(\alpha_{n+i}+l_i+\sum_{\substack{j=1\\ j\neq i}}^{n} h_j-k_i\right)!}\times\right.$$

$$\times\left[c_i t-\mu_i\frac{\cos(a_i t+\omega_i)}{a_i}+\mu_i\frac{\cos\omega_i}{a_i}\right]^{\alpha_{n+i}+l_i+\sum_{i=1}^{n}h_i-k_i}\times$$

$$\times\left[\left(e_i t-\gamma_i\frac{\cos(\eta_i t+\delta_i)}{\eta_i}+\gamma_i\frac{\cos\delta_i}{\eta_i}\right)\left(1-u(\alpha_i+q_i-g_i)\right)\right]^{q_i}\times$$

$$\times\left[\left(\rho_i t-\beta_i\frac{\cos(\theta_i t+\nu_i)}{\theta_i}+\beta_i\frac{\cos\nu_i}{\theta_i}\right)u(\alpha_i+q_i-g_i)\right]^{g_i}\Bigg] \quad (16)$$

Example 2. Let $\lambda=40$, $a=1/2$, $\omega=1$, $b=6$, $\mu_1=\mu_3=2$, $\mu_2=\mu_4=8$, $a_1=a_2=1.5$, $a_3=a_4=2$, $\omega_1=\omega_2=\omega_3=\omega_4=0.1$, $c_1=c_2=3$, $c_3=c_4=-2$, $\beta_1=1.7$, $\beta_2=0.5$, $\beta_3=\beta_4=1$, $\theta_1=\theta_2=\theta_3=\theta_4=0.1$, $\nu_1=\nu_2=\nu_3=\nu_4=0.6$, $\rho_1=\rho_2=\rho_3=\rho_4=5$, $\gamma_1=2.3$, $\gamma_2=\gamma_3=\gamma_4=3.5$, $\eta_1=\eta_2=\eta_3=\eta_4=1$, $\delta_1=\delta_2=\delta_3=\delta_4=0$, $e_1=e_2=e_3=e_4=0$, $p_{0i}=1/4$, $i=\overline{1,4}$, $p_{i0}=2/5$, $i=\overline{1,3}$, $p_{40}=1/2$, $p_{4i}=1/6$, $i=\overline{1,3}$, $p_{i4}=3/5$, $i=\overline{1,3}$, $p_{ii}=0$, $i=\overline{0,4}$. In Figure 2, the plot of state probability $P(0,1,1,1,6,2,7,10,t)$ is represented, if QN at the initial moment is in state $(1,0,0,1,3,8,6,4)$

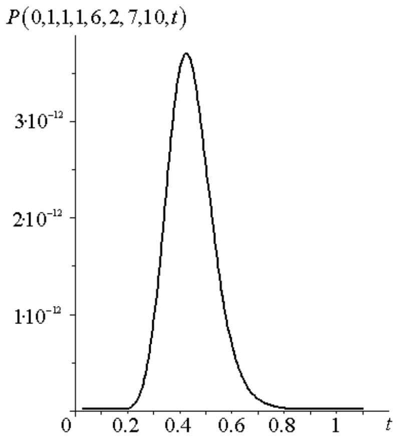

Fig. 2. Plot of probability $P(0,1,1,1,6,2,7,10,t)$

Conclusion

In this article the method of generating functions is observed. This method is used for the investigation of a queuieng network with unreliable systems of arbitrary topology in a transient regime and condition of heavy loading. Such networks can serve as models of the functioning of LAN. Expressions for the network state probabilities are obtained. Furthermore it is supposed to receive expressions for time-dependent average characteristics of the network.

References

[1] Matalytski M., Queueing networks at stationary and transient regime, Monograph, GrSU, Grodno 2001 (in Russian).

[2] Vishnevski V., Theory of computer network design, Moscow 2003 (in Russian).