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DETERMINISTIC FRACTALS BASED ON ARCHIMEDEAN SOLIDS

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Abstract. In the present work, the construction of fractals based on Archimedean solids was discussed. The methods of 3D fractals construction based on uniform polyhedra were presented. It was shown that the contraction mapping procedure for the construction of fractals with non-overlapping or -disjointed contractions could be applied only for a limited number of the polyhedra. The contraction ratios and the Hausdorff dimensions were determined for the existing fractals with adjacent contractions based on Archimedean solids.

Introduction

The development of fractal geometry created new ways of solving topical scientific problems from quantum physics [1] through structural diagnostics [2] to biology and genetics [3]. They found applications in problems related to the mechanics of porous media or in modeling the rheology of materials. However, the most intensive development of the application of fractals was noticed in the problems of computer graphics, pattern recognition, image compression and coding [4]. Deterministic fractals are used in testing ray-tracing algorithms during the rendering of spatial scenes [5].

As is well known, the construction of fractals is based on the self-similarity of the input object. Two general approaches are used in the construction process: random and deterministic ones. The random approach usually uses the well-known Barnsley's Chaos Game or its combination with the Iterated Function System (IFS), e.g. Chaos Game for IFS connected in the net [6]. The deterministic approach is based usually on the Multiple Reduction Copy Machine (MRCM) or on the IFS. Some of the simplest and earlier deterministic fractals are the Sierpiński triangle and Sierpiński carpet. The generalization of Sierpiński fractals based on *N*-sided polygons was presented in [7]. The Sierpiński triangle and carpet were generalized to the three-dimensional space and known as the Sierpiński tetrahedron and Menger sponge. Only these two 3D fractals are well-known and presented in many literature positions, e.g. [6, 8]. In [9] the authors determined the contraction ratios and Hausdorff dimensions for fractals based on Platonic solids, however, they did not give any rules for the construction of such fractals.

1. General considerations

Because of the regularity of Platonic solids, fractals based on them could be constructed as well. However, there is a group of uniform or even semi-regular polyhedra which cannot be used construct non-overlapped or - disjointed fractals. It could be presented during the construction of fractals based on Archimedean solids. Further they will be called *Archimedean fractals*, which could be defined as follows.

Definition. Let A_0^S be an Archimedean solid with the set of vectors v_n which represents the vertices with coordinates $v_{n,a}$, $\mathrm{a}=1,2,3$, in the Euclidean space $\mathbf{R}^3$, where S denotes the Schläfli symbol of the given polyhedron and the number in the subscript of A denotes the number of contraction mapping iterations. Then the fractal based on the given Archimedean solid could be defined as attractor A_∞^S of the IFS, which is the set of

$$A_\infty^S = \bigcap_{i=0}^{\infty} w_i\left(A_0^S\right), \tag{1}$$

where the contraction process of A_k to A_{k+1} was realized with use of the Hutchinson operator:

$$W\left(A_{k+1}^S\right) = \bigcup_{i=1}^{N} w_i\left(A_k^S\right). \tag{2}$$

Here $w_i(\bullet)$ is an elementary similarity transformation and N denotes the number of contractions in a given subset, thus

$$\forall_v \quad w_i\left(v_n\right) = \frac{v_n}{r(S)} - \frac{v_{n,a}\left(1-r(S)\right)}{r(S)}, \tag{3}$$

where $r(S)$ is the unique contraction factor for polyhedron S which ensures that the contractions of w_i are non-overlapped or disjointed. That is, $w_i\left(A_k^S\right) \cap w_j\left(A_k^S\right) = \varnothing$ for $i \neq j$. Finally, using (2) the attractor of A_0^S could be obtained in the form:

$$A_\infty^S = \bigcup_{i=1}^{\infty} W^i\left(A_0^S\right), \tag{4}$$

while $W^0\left(A_0^S\right) = A_0^S$. It implies the following: $A_\infty^S = \lim\limits_{k\to\infty} A_k^S$.

Fractals are characterized by a fractional dimension in general (except some fractals like the Sierpiński tetrahedron or Hilbert cube). There are many formulations of fractal measures including box dimension, capacity dimension, topological dimension, Lebesgue dimension, Minkowski dimension etc. [6, 8], which often are incorrectly used as synonyms or generalized as a fractal dimension. The most general formulation of the fractal dimension was presented by Hausdorff [10], which is a power law such as:

$$N = \frac{1}{r^D} \ \text{or} \ D = \frac{\ln(N)}{\ln(r)} \tag{5}$$

In the case of most 3D fractals, $2 \geq D \geq 3$ (with some exceptions, e.g. Cantor dust with $D \approx 1.89$).

The aim of this work is to determine the method of construction of fractals based on convex uniform polyhedra, which is possible by the analysis of Archimedean solids, and determine the contraction ratios and Hausdorff dimensions for the fractals defined above.

2. Construction of Archimedean fractals

Let us consider Archimedean solid A_0^S with vertices $v_n \in \mathbf{R}^3$ inscribed in sphere $P \in \mathbf{R}^3$ of unit radius R with the central point of $c = [0,0,0]$. Applying contraction mapping procedure (2), one obtains n contractions scaled $1/r$ times, where one of the vertices of the given contraction coincides with one of the vertices of A_0^S.

Consider the fist example - one of the simplest Archimedean solids - the truncated tetrahedron, $A_0^{t[3,3]}$, which contains 4 triangular and 4 hexagonal faces. The truncation was realized from the tetrahedron, thus a similar principle as for the Sierpiński tetrahedron could be used. To determine the contraction ratio for $A_\infty^{t[3,3]}$, only the hexagonal face could be considered: r will be the same as for the Sierpiński hexagon. It could be calculated from the formula (cf. [7]):

$$r = \frac{1}{2}\left(\frac{\tan\frac{\pi}{m}\operatorname{int}\left(\frac{m-1}{4}\right)}{\tan\frac{\pi}{m} + \frac{\pi}{m}\operatorname{int}\left(\frac{m-1}{4}\right)} \right) \tag{6}$$

where m is the number of vertices of a regular polygon, $m \geq 5$. Having the contraction ratio and coordinates of the vertices of the polyhedron [11], it is possible to determine the central points of circum-spheres for $A_1^{t[3,3]}$ and the next iterations and place in those central points the contractions scaled by $1/r$.

Proposition. Let $w_i\left(A_k^S\right)$ be a finite number of contractions of A_k^S in terms of the Definition. Then central points $c_{i,k}$ of the contractions could be determined recursively from the vertices of A_k^S.

Proof. Because A_0^S is a semi-regular polyhedron, it can be inscribed into a sphere with radius R and $v_n \in P$. Considering the strict self-similarity between A_k^S and A_{k+1}^S and the fact that $r(S)$ is unique, one could conclude that

$$c_{n,k}\left(A_{k+1}^S\right)=\frac{r(S)-R}{r(S)^k}v_n\left(A_k^S\right), \tag{7}$$

which ends the proof.

Truncated tetrahedron $A_0^{t[3,3]}$ and the two first approximations of the truncated octahedron fractal are presented in Figure 1.

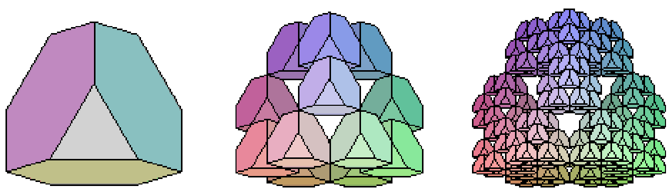

Fig. 1. Construction of truncated tetrahedron fractal

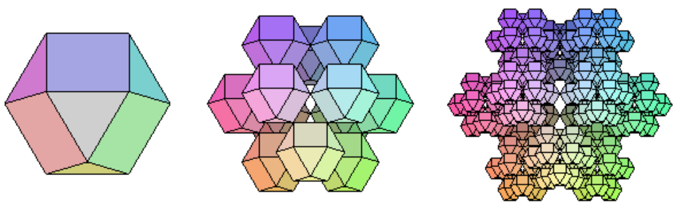

Fig. 2. Construction of cuboctahedron fractal

Similarly, it is possible to construct fractals for other Archimedean solids: cuboctahedron $A_0^{t_{0,2}[3,3]}$ and small rhombicuboctahedron $A_0^{t_{0,2}[4,3]}$, which are presented in Figures 2 and 3, respectively. The contraction ratios for these fractals were calculated based on the cross-sections of the polyhedra by the plane through the edges. In the first case, one obtained a hexagon $(\mathrm{r}=3)$ and in the second case - an octagon $(\mathrm{r}=2+\sqrt{2})$.

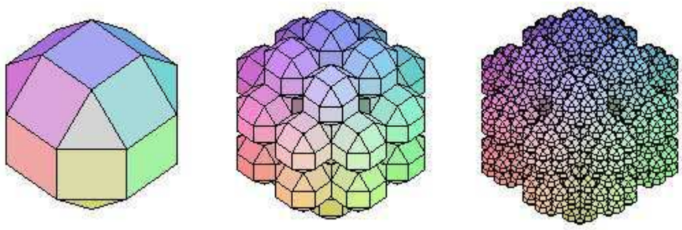

Fig. 3. Construction of the small rhombicuboctahedron fractal

The above-presented two methods of fractals construction could not be applied to other, more complex Archimedean solids, thus the next method should be introduced.

Let $A_0^{t_{0,1}[3,4]}$, be the truncated octahedron inscribed in the circum-sphere of a unit radius with the central point of $c=[0,0,0]$. For the construction of the first iteration of $A_\infty^{t_{0,1}[3,4]}$ one chooses one of its faces as the base (in this case the hexagonal one). Considering the above-mentioned considerations, six of its contractions will be placed on the base (see Fig. 4). Taking into account the strict self-similarity between any two iterations, the contraction ratio could be determined as a ratio of the side lengths s_k/s_{k+1} of A_k^S and A_{k+1}^S (Fig. 5).

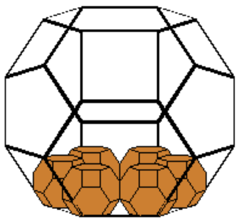

Fig. 4. Contractions

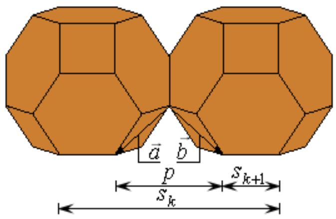

Fig. 5. Dimensions of contractions

Having the coordinates of the vertices, one determined vectors $\vec{a}$ and $\vec{b}$ and the angle between them for determining p. Now, the side length of the contraction could be determined as $s_{k+1}=(s_k-p)/2$.

Based on the above presented method, the following fractals could be constructed: truncated octahedron fractal $A_0^{t_{0,1}[3,4]}$, icosidodecahedron fractal $A_0^{t_1[5,3]}$, truncated dodecahedron fractal $A_0^{t[5,3]}$, truncated icosahedron fractal $A_0^{t[3,5]}$, small rhombiicosidodecahedron fractal $A_0^{t_{0,2}[5,3]}$, great rhombiicosidodecahedron fractal $A_0^{t_{0,1,2}[5,3]}$ and the fractals presented before as well. Their construction is presented in Figures 6-11, respectively.

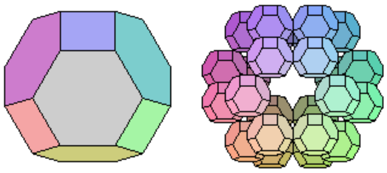

Fig. 6. Construction of truncated octahedron fractal

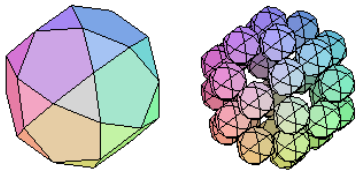

Fig. 7. Construction of icosidodecahedron fractal

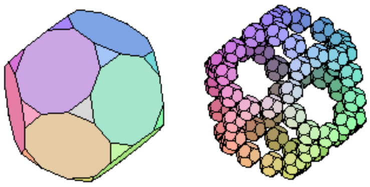

Fig. 8. Construction of truncated dodecahedron fractal

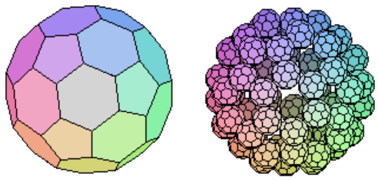

Fig. 9. Construction of truncated icosahedron fractal

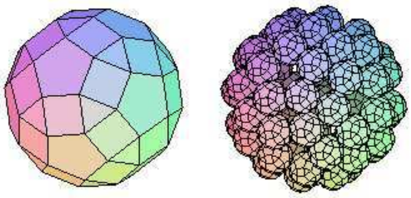

Fig. 10. Construction of small rhombiicosidodecahedron fractal

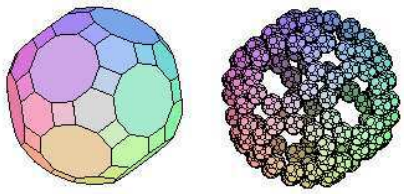

Fig. 11. Construction of great rhombiicosidodecahedron fractal

Table 1

Contraction ratios and Hausdorff dimensions for existing Archimedean fractals

Schlafli symbol, S	Contraction ratio, r	Hausdorff dimension, D
$t[3,3]$	3	2.2618
$t_{0,2}[3,3]$	3	2.2618
$t_{0,2}[4,3]$	$2+\sqrt{2}$	2.5882
$t_{0,1}[3,4]$	4	2.2925
$t_{1}[5,3]$	4.2361	2.3561
$t[5,3]$	6.8538	2.1271
$t[3,5]$	5.8544	2.3168
$t_{0,2}[5,3]$	$\left(5-\sqrt{5}\right)/\left(5-2\sqrt{5}\right)$	2.4729
$t_{0,1,2}[5,3]$	$2\sqrt{5}/\left(5-2\sqrt{5}\right)$	2.2404

The Hausdorff dimensions for the presented fractals were calculated and tabulated in Table 1 together with the contraction ratios. In the cases of $t_1[5,3]$, $t[5,3]$ and $t[3,5]$, the approximate values of the contraction ratios were given because of the high complexity of the form of their exact values.

3. Limitations of fractals construction based on Archimedean solids

In Section 2, nine of thirteen Archimedean solids were discussed and the fractals based on them were defined. The last four Archimedean solids, which are uniform and semi-regular, are impossible for fractal construction.

Theorem. Let A_0^S be an Archimedean solid with vertices $v_n \in \mathbf{R}^3$. Considering the definition, the Archimedean fractal based on A_0^S could be constructed if and only if the unique contraction ratio exists.

Proof. Following the presented method of fractals construction (see Sec. 1), the ratio between s_k and s_{k+1} must be the same regardless of the chosen base. For instance, let us consider the application of a construction method to the truncated hexahedron $A_0^{t[4,3]}$. As was presented in Figure12, there are two possible bases: a triangular and octagonal. Since points $v_n\left(A_0^S\right)$ are the prisoner points of A_∞^S (see [12] p. 74 for the definition), there is a relation between the contractions for arbitrary iterations: $v_n\left(A_k^S\right) = v_{n,n}\left(A_{k+1}^S\right)$, thus $w_i\left(A_k^S\right)$ has exactly one common point with A_k^S. Considering that all of the edges of A_k^S are equal: $dist(B,C) = dist(C,D)$. However, $\mathrm{proj}(\mathrm{dist}(A,B)) \neq \mathrm{proj}(\mathrm{dist}(F,C))$ and $\mathrm{proj}(\mathrm{dist}(E,C)) \neq \mathrm{proj}(\mathrm{dist}(E,D))$. In such a situation one obtains multiple values of r_q, where q is the number of polygons types from which the polyhedron is composed. The contractions scaled by $1/r_q$ are overlapped or disjointed or overlapped and disjointed simultaneously (Fig. 13), which is in conflict with the definition of an Archimedean fractal.

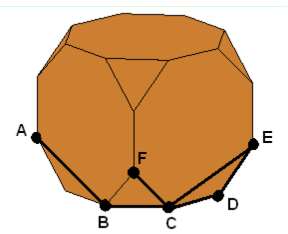

Fig. 12. Vertices of interest of truncated hexahedron

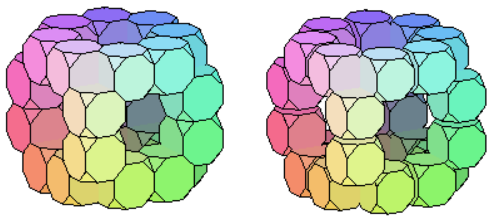

Fig. 13. Examples of overlapped $(r_1 = 2+\sqrt{2})$ and disjointed $(r_2 \approx 3.8476)$ contractions of truncated hexahedron

Similarly, the great rhombicuboctahedron $A_0^{t_{0,1,2}[4,3]}$, the snub hexahedron $A_0^{s[4,3]}$ and the snub dodecahedron $A_0^{s[5,3]}$ do not fulfill the Definition either and $A_\infty^{t_{0,1,2}[4,3]}$, $A_\infty^{s[4,3]}$, $A_\infty^{s[5,3]}$ are not Archimedean fractals.

Conclusions

Archimedean fractals were introduced and the methods of their construction were proposed. The contraction ratios and Hausdorff dimensions were determined for existing Archimedean fractals. It was proven that not every Archimedean solid is able to construct a fractal with adjacent contractions. The presented methods of the fractal construction and the theorem could be extended to other uniform polyhedra.

References

[1] Maker D., Quantum physics and fractal space time, Chaos Soliton. Fract. 1999, 10, 31-42.
[2] Hadjileontiadis L.J., Douka E., Trochidis A., Fractal dimension analysis for crack identification in beam structures, Mech. Syst. Signal Pr. 2005, 19, 659-674.
[3] Auffray C., Nottale L., Scale relativity theory and integrative systems biology: 1 Founding principles and scale laws, Prog. Biophys. Mol. Bio. 2008, 97, 79-114.
[4] Ozawa K., Dual fractals, Image Vision Comput. 2008, 26, 622-631.
[5] Hart J.C., Sandin D.J., Kauffman, L.H., Ray tracing deterministic 3D fractals, Proceedings of the 16th Annual Conference on Computer Graphics and interactive Techniques SIGGRAPH '89, ACM Press, New York 1989, 289-296.
[6] Peitgen H.-O., Jürgens H., Saupe D., Fractals for the classroom. Part I: Introduction to fractals and chaos, Springer-Verlag, New York 1992.
[7] Kahng B., Davis J., Maximal dimensions of uniform Sierpinski fractals, Fractals 2010, 18, 451-460.
[8] Kudrewicz J., Fractals and chaos, WNT, Warsaw 2007 (in Polish).
[9] Kunnen A., Schlicker S., Regular Sierpinski polyhedra, Pi Mu Epsilon J. 1998, 10, 607-619.
[10] Hausdorff F., Dimension und äußeres Maß, Math. Ann. 1918, 79, 157-159.
[11] Coxeter H.S.M., Regular Polytopes, Courier Dover Publications, New York 1973.
[12] Peitgen H.-O., Jürgens H., Saupe D., Chaos and Fractals: New Frontiers of Science, 2nd ed., Springer-Verlag, New York 2004.