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APPLICATION OF BEM FOR NUMERICAL SOLUTION OF THERMAL WAVE MODEL OF BIOHEAT TRANSFER

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Abstract. The thermal wave model of bioheat transfer supplemented by boundary and initial conditions is considered. To solve the problem, the boundary element method (BEM) is proposed. In the final part of the paper examples of numerical computations concerning the determination of the temperature field in a heating tissue are shown.

Introduction

Heat transfer in living tissues, subjected to the action of strong external heat sources can be described using different mathematical models. The most popular is the Pennes equation [1-5] based on classical Fourier law. According to the newest opinions [6-9], heat conduction proceeding in the biological tissue domain should be described by using a hyperbolic equation (Cattaneo-Vernotte equation [10, 11]) in order to take into account its nonhomogeneous inner structure. In the paper, the method of solving the Cattaneo-Vernotte equation for a 2D problem is proposed. It is the boundary element method using discretization in time adapted for the numerical solution of the thermal wave equation. In successive chapters, the boundary integral equation is derived, the numerical model is described and the results of computations are shown.

1. Thermal wave equation

The thermal wave model of bioheat transfer in living tissues is the following [7, 8]:

$$c\left[\tau\frac{\partial^2 T(x,t)}{\partial t^2}+\frac{\partial T(x,t)}{\partial t}\right]=\lambda\nabla^2 T(x,t)+Q(x,t)+\tau\frac{\partial Q(x,t)}{\partial t}, \quad (1)$$

where c, λ denote the volumetric specific heat and thermal conductivity of tissue, respectively, $Q(x, t)$ is the volumetric heat due to metabolism and blood perfusion,

$\tau = a/C^2$ is the relaxation time, ($a = \lambda/c$ is the diffusion coefficient, C is the velocity of thermal wave), T is the tissue temperature, x, t denote the spatial co-ordinates and time. Function $Q\,(x, t)$ is equal to

$$Q(x,t) = G_B\, c_B \left[T_B - T(x,t) \right] + Q_m\,, \tag{2}$$

where G_B is the blood perfusion rate, c_B is the volumetric specific heat of blood, T_B is the artery temperature and Q_m is the metabolic heat source.

It should be pointed out that for $\tau = 0$, equation (1) reduces to the well-known Pennes bioheat equation.

Equation (1) is supplemented by boundary conditions

$$\begin{aligned} x \in \Gamma_1 :\quad & T(x,t) = T_b(x) \\ x \in \Gamma_2 :\quad & q(x,t) = q_b(x) \end{aligned} \tag{3}$$

and initial ones

$$t = 0:\quad T(x,t) = T_0\,, \quad \left.\frac{\partial T(x,t)}{\partial t}\right|_{t=0} = 0\,, \tag{4}$$

where $q(x, t)$ is the boundary heat flux, $T_b(x)$, $q_b(x)$ are the known boundary temperature and boundary heat flux and T_0 is known initial temperature of biological tissue.

Taking into account formula (2), equation (1) can be written in the form

$$\tau \frac{\partial^2 T(x,t)}{\partial t^2} + \frac{\partial T(x,t)}{\partial t} = a\nabla^2 T(x,t) + \frac{k}{c}\left[T_B - T(x,t) \right] + \frac{Q_m}{c} - \frac{\tau k}{c}\frac{\partial T(x,t)}{\partial t} \tag{5}$$

or

$$\tau \frac{\partial^2 T(x,t)}{\partial t^2} + \left(1 + \frac{\tau k}{c}\right)\frac{\partial T(x,t)}{\partial t} = a\nabla^2 T(x,t) + \frac{k}{c}\left[T_B - T(x,t) \right] + \frac{Q_m}{c} \tag{6}$$

where $k = G_B c_B$.

2. Boundary element method using discretization in time

To solve equation (6), the BEM using discretization in time is applied [12, 13]. At first, the time grid with constant step $\Delta t = t^f - t^{f-1}$ is introduced.

Using the Lagrange interpolation for points (t^{f-2}, T^{f-2}), (t^{f-1}, T^{f-1}), (t^f, T^f), where $T^{f-2} = T(x, t^{f-2})$, $T^{f-1} = T(x, t^{f-1})$, $T^f = T(x, t^f)$, one obtains

$$
\begin{gathered}
T\left(x,t\right)=T^{f-2}\,\frac{\left(t-t^{f-1}\right)\left(t-t^{f}\right)}{2\left(\Delta t\right)^{2}}-\\
T^{f-1}\,\frac{\left(t-t^{f-2}\right)\left(t-t^{f}\right)}{\left(\Delta t\right)^{2}}+T^{f}\,\frac{\left(t-t^{f-2}\right)\left(t-t^{f-1}\right)}{2\left(\Delta t\right)^{2}}
\end{gathered}
\tag{7}
$$

On the basis of (7), the time derivatives are calculated and then

$$
\left.\frac{\partial T\left(x,t\right)}{\partial t}\right|_{t=t^{f}}=\frac{T^{f-2}-4T^{f-1}+3T^{f}}{2\Delta t}
\tag{8}
$$

while

$$
\left.\frac{\partial^{2} T\left(x,t\right)}{\partial t^{2}}\right|_{t=t^{f}}=\frac{T^{f-2}-2T^{f-1}+T^{f}}{\left(\Delta t\right)^{2}}.
\tag{9}
$$

Taking into account formulas (8), (9), the following approximation of equation (6) is obtained

$$
\begin{gathered}
\tau\,\frac{T\left(x,t^{f-2}\right)-2T\left(x,t^{f-1}\right)+T\left(x,t^{f}\right)}{\left(\Delta t\right)^{2}}+\\
\left(1+\frac{\tau k}{c}\right)\frac{T\left(x,t^{f-2}\right)-4T\left(x,t^{f-1}\right)+3T\left(x,t^{f}\right)}{2\Delta t}=\\
a\nabla^{2}\,T\left(x,t^{f}\right)+\frac{k}{c}\left[T_{B}-T\left(x,t^{f}\right)\right]+\frac{Q_{m}}{c}
\end{gathered}
\tag{10}
$$

or

$$
\nabla^{2}T\left(x,t^{f}\right)-AT\left(x,t^{f}\right)+BT\left(x,t^{f-1}\right)-CT\left(x,t^{f-2}\right)+\frac{Q}{\lambda}=0,
\tag{11}
$$

where

$$
\begin{gathered}
A=\frac{\tau}{a\left(\Delta t\right)^{2}}+\frac{3\left(c+\tau k\right)}{2\lambda\Delta t}+\frac{k}{\lambda},\quad B=\frac{2\tau}{a\left(\Delta t\right)^{2}}+\frac{2\left(c+\tau k\right)}{\lambda\Delta t}\\
C=\frac{\tau}{a\left(\Delta t\right)^{2}}+\frac{c+\tau k}{2\lambda\Delta t},\quad Q=kT_{B}+Q_{m}
\end{gathered}\,.
\tag{12}
$$

For equation (11) the weighted residual criterion is applied

$$\iint_{\Omega} \left[\nabla^2 T\left(x, t^f\right) - AT\left(x, t^f\right) + BT\left(x, t^{f-1}\right) - CT\left(x, t^{f-2}\right) + \frac{Q}{\lambda} \right] T^*(\xi, x)\, \mathrm{d}\Omega = 0 \tag{13}$$

where ξ is the observation point and $T^*(\xi,x)$ is the fundamental solution and this function should fulfil the equation

$$\nabla^2 T^*(\xi, x) - AT^*(\xi, x) = -\delta(\xi, x) \tag{14}$$

where $\delta(\xi, x)$ is the Dirac function.

For a 2D problem and domain oriented in the Cartesian co-ordinate system it is the following function

$$T^*(\xi, x) = \frac{1}{2\pi} \mathrm{K}_0\left(r\sqrt{A}\right), \tag{15}$$

where $\mathrm{K}_0(\cdot)$ is the modified Bessel function of the second kind of zero order [12, 13], r is the distance between observation point $\xi = (\xi_1, \xi_2)$ and point $x = (x_1, x_2)$.

Applying the 2nd Green formula for the first component of equation (13), one obtains

$$\iint_{\Omega} \nabla^2 T\left(x, t^f\right) T^*(\xi, x)\, \mathrm{d}\Omega = \iint_{\Omega} \nabla^2 T^*(\xi, x)\, T\left(x, t^f\right) \mathrm{d}\Omega + \int_{\Gamma} \left[T^*(\xi, x)\, \mathbf{n} \cdot \nabla T\left(x, t^f\right) - T\left(x, t^f\right) \mathbf{n} \cdot \nabla T^*(\xi, x) \right] \mathrm{d}\Gamma \tag{16}$$

and then criterion (13) takes the form

$$\begin{aligned} &\iint_{\Omega} \left[\nabla^2 T^*(\xi, x) - AT^*(\xi, x) \right] T\left(x, t^f\right) \mathrm{d}\Omega + \\ &\iint_{\Omega} \left[BT\left(x, t^{f-1}\right) - CT\left(x, t^{f-2}\right) + \frac{Q}{\lambda} \right] T^*(\xi, x)\, \mathrm{d}\Omega + \\ &\int_{\Gamma} \left[T^*(\xi, x)\, \mathbf{n} \cdot \nabla T\left(x, t^f\right) - T\left(x, t^f\right) \mathbf{n} \cdot \nabla T^*(\xi, x) \right] \mathrm{d}\Gamma = 0 \end{aligned} \tag{17}$$

Using property (14) of a fundamental solution, one has

$$\begin{aligned} T\left(\xi, t^f\right) + \frac{1}{\lambda} \int_{\Gamma} T^*(\xi, x)\, q\left(x, t^f + \tau\right) \mathrm{d}\Gamma = \frac{1}{\lambda} \int_{\Gamma} q^*(\xi, x)\, T\left(x, t^f\right) \mathrm{d}\Gamma + \\ \iint_{\Omega} \left[BT\left(x, t^{f-1}\right) - CT\left(x, t^{f-2}\right) + \frac{Q}{\lambda} \right] T^*(\xi, x)\, \mathrm{d}\Omega \end{aligned} \tag{18}$$

where

$$q\left(x, t^f+\tau\right)=-\lambda\,\mathbf{n}\cdot\nabla T\left(x, t^f\right), \quad q^*\left(\xi, x\right)=-\lambda\,\mathbf{n}\cdot\nabla T^*\left(\xi, x\right) \tag{19}$$

Function $q^*(\xi, x)$ can be calculated analytically and then

$$q^*\left(\xi, x\right)=\frac{\lambda d\sqrt{A}}{2\pi r}\,\mathrm{K}_1\left(r\sqrt{A}\right), \tag{20}$$

where $\mathrm{K}_1(\cdot)$ is the modified Bessel function of the second kind of first order [12, 13], while

$$d=\left(x_1-\xi_1\right)n_1+\left(x_2-\xi_2\right)n_2 \tag{21}$$

For $\xi\in\Gamma$, one obtains the following boundary integral equation

$$\begin{gathered} B(\xi)\,T\left(\xi, t^f\right)+\frac{1}{\lambda}\int_{\Gamma}T^*\left(\xi, x\right)q\left(x, t^f+\tau\right)\mathrm{d}\Gamma= \\ \frac{1}{\lambda}\int_{\Gamma}q^*\left(\xi, x\right)T\left(x, t^f\right)\mathrm{d}\Gamma+\iint_{\Omega}\left[BT\left(x, t^{f-1}\right)-CT\left(x, t^{f-2}\right)+\frac{Q}{\lambda}\right]T^*\left(\xi, x\right)\mathrm{d}\Omega \end{gathered} \tag{22}$$

where $B(\xi)\in(0, 1)$. The value of coefficient $B(\xi)$ results from the position of boundary point ξ considered, for example for the smooth fragment of boundary $B(\xi)=0.5$. Equation (22) constitutes a basis for numerical algorithm construction.

3. Numerical realization

To solve equation (22), boundary Γ is divided into N boundary elements and interior Ω is divided into L internal cells. Hence, the approximate form of equation (22) is the following:

$$\begin{gathered} B\left(\xi^i\right)T\left(\xi^i, t^f\right)+\frac{1}{\lambda}\sum_{j=1}^{N}\int_{\Gamma_j}T^*\left(\xi^i, x\right)q\left(x, t^f+\tau\right)\mathrm{d}\Gamma_j= \\ \frac{1}{\lambda}\sum_{j=1}^{N}\int_{\Gamma_j}q^*\left(\xi^i, x\right)T\left(x, t^f\right)\mathrm{d}\Gamma_j+ \\ \sum_{l=1}^{L}\iint_{\Omega_l}\left[BT\left(x, t^{f-1}\right)-CT\left(x, t^{f-2}\right)+\frac{Q}{\lambda}\right]T^*\left(\xi^i, x\right)\mathrm{d}\Omega_l \end{gathered} \tag{23}$$

For constant boundary elements and constant internal cells [12, 13] one has

$$
\begin{gathered}
\frac{1}{2} T\left(\xi^{i}, t^{f}\right)+\frac{1}{\lambda} \sum_{j=1}^{N} q\left(x^{j}, t^{f}+\tau\right) \int_{\Gamma_{j}} T^{*}\left(\xi^{i}, x\right) \mathrm{d} \Gamma_{j}= \\
\frac{1}{\lambda} \sum_{j=1}^{N} T\left(x^{j}, t^{f}\right) \int_{\Gamma_{j}} q^{*}\left(\xi^{i}, x\right) \mathrm{d} \Gamma_{j}+ \\
\sum_{l=1}^{L}\left[B T\left(x^{l}, t^{f-1}\right)-C T\left(x^{l}, t^{f-2}\right)+\frac{Q}{\lambda}\right] \iint_{\Omega_{l}} T^{*}\left(\xi^{i}, x\right) \mathrm{d} \Omega_{l}
\end{gathered} \tag{24}
$$

The following notations are introduced

$$
G_{ij}=\frac{1}{\lambda} \int_{\Gamma_{j}} T^{*}\left(\xi^{i}, x\right) \mathrm{d} \Gamma_{j}, \quad \hat{H}_{ij}=\frac{1}{\lambda} \int_{\Gamma_{j}} q^{*}\left(\xi^{i}, x\right) \mathrm{d} \Gamma_{j} \tag{25}
$$

$$
P_{il}=\iint_{\Omega_{l}} T^{*}\left(\xi^{i}, x\right) \mathrm{d} \Omega_{l} \tag{26}
$$

and then equation (24) takes the form

$$
\frac{1}{2} T_{i}^{f}+\sum_{j=1}^{N} G_{ij} q_{j}^{f}=\sum_{j=1}^{N} \hat{H}_{ij} T_{j}^{f}+\sum_{l=1}^{L} P_{il}\left(B T_{l}^{f-1}-C T_{l}^{f-2}+\frac{Q}{\lambda}\right) \tag{27}
$$

or

$$
\sum_{j=1}^{N} G_{ij} q_{j}^{f}=\sum_{j=1}^{N} H_{ij} T_{j}^{f}+\sum_{l=1}^{L} P_{il}\left(B T_{l}^{f-1}-C T_{l}^{f-2}+\frac{Q}{\lambda}\right), \tag{28}
$$

where

$$
H_{ij}=\begin{cases}\hat{H}_{ij}, & i \neq j \\ \hat{H}_{ii}-\dfrac{1}{2}, & i=j\end{cases} \tag{29}
$$

Equations (28) written for all boundary nodes $i = 1, 2, \ldots, N$ create the system of N algebraic equations which can be written in the matrix form

$$
\mathbf{G} \mathbf{q}^{f}=\mathbf{H}\, \mathbf{T}^{f}+\mathbf{P}\left(B \mathbf{T}^{f-1}-C \mathbf{T}^{f-2}+\frac{Q}{\lambda}\right) \tag{30}
$$

This system of equations allows one to determine the ‘missing’ boundary temperatures and boundary heat fluxes.

Next, the temperatures at the internal nodes $\xi^i \in \Omega$, $i = N + 1, N + 2, \ldots, N + L$ are calculated using the formula

$$T_i^f = \sum_{j=1}^{N} H_{ij} T_j^f - \sum_{j=1}^{N} G_{ij} q_j^f + \sum_{l=1}^{L} P_{il} \left(B T_l^{f-1} - C T_l^{f-2} + \frac{Q}{\lambda} \right) . \tag{31}$$

4. Results of computations

The biological tissue domain of dimensions 0.015 m × 0.015 m (L = 0.015 m) has been considered. The initial temperature of the tissue equals T_0 = 37°C. The following input data have been taken into account: λ = 0.75 W/(mK), $c = 3\cdot 10^6$ W/(m^3 K), G_B = 0.0005 1/s, $c_B = 3.9962\cdot 10^6$ W/(m^3 K), T_B = 37°C, Q_m = 245 W/m^3.

On boundary $x_1 = 0$, $0 \le x_2 \le L$, the Dirichlet condition in the form of $T_b(x_2) = T_{max} + ((T_0 - T_{max}) x_2 / L)$ has been assumed, on boundary $x_1 = L$, $0 \le x_2 \le L$, temperature T = 37°C has been accepted, while on the remaining part of the boundary, the no-flux condition has been assumed.

The boundary has been divided into 60 constant boundary elements, the interior has been divided into 225 constant internal cells. Time step: Δt = 20 s.

Figures 1, 2 illustrate the heating curves at points 1 - (0.0055, 0.0055), 2 - (0.0095, 0.0095) and 3 - (0.0135, 0.0135).

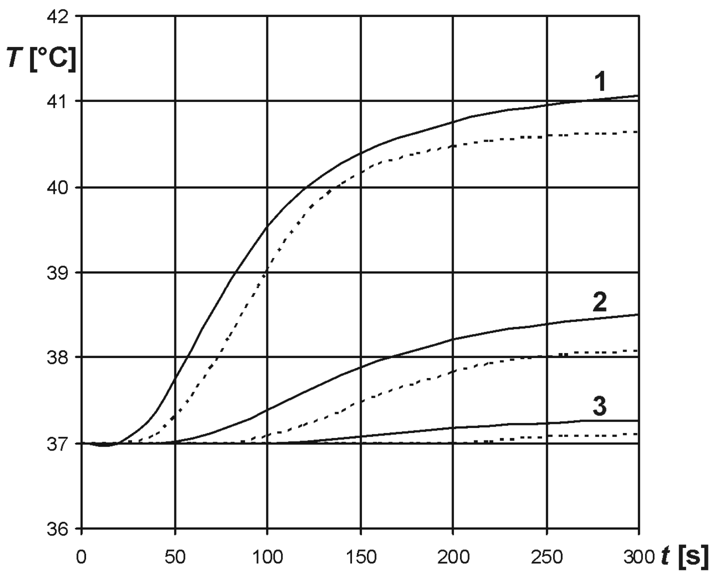

Fig. 1. Heating curves at points 1, 2 and 3
for T_{max} = 50°C (full lines - τ = 0 s, broken lines - τ = 20 s)

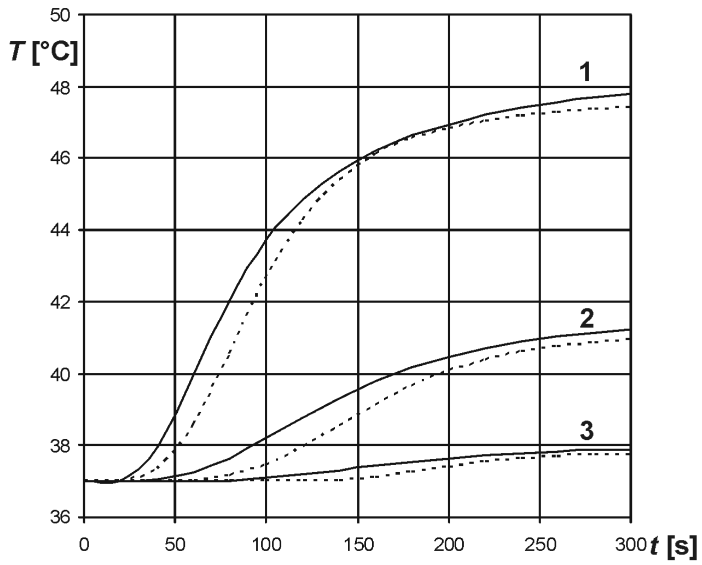

Fig. 2. Heating curves at points 1, 2 and 3
for T_{max} = 70°C (full lines - τ = 0 s, broken lines - τ = 20 s)

Conclusions

The 2D thermal wave equation has been solved by means of the boundary element method. Under the assumption that $\tau = 0$, the results of computations have been compared to the results obtained for the Pennes equation using the classical boundary element method and they confirm the effectiveness and exactness of the proposed algorithm.

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