

## MATHEMATICAL DESCRIPTION OF DISCONTINUOUS GALERKIN METHOD IN THE THEORY OF THERMOELASTICITY

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**Abstract.** In the paper mathematical description of Discontinuous Galerkin Method (DGM) used in the theory of thermoelasticity is presented. Displacement form of governing equations is introduced as the base of mathematical model. Space discretization methodology for discontinuous finite element method is showed.

### Introduction

DGM is in fact similar to classical FEM (Finite Element Method). It makes use of the same function space, but continuity at the element boundaries is relaxed. It was first introduced by Reed and Hill [1] in 1973 for the solution of nuclear transport equation. Cockburn [2, 3] has restored the popularity of DGM at the beginning of this century. The main idea of the method is that the field variable, its derivatives or even both of them are considered discontinuous across the element boundaries. From this point of view DGM includes both the FEM and FDM (Finite Difference Method) thus it has advantages of both of them. DGM is especially useful for computational dynamics and heat transfer. Because of the locality of the discontinuous approximation, there is no need to build global matrix and thus this reduces the demand of RAM. The influence of the boundary conditions is propagated gradually through the mesh. DGM is currently widely discussed in the literature particularly in relation to convection-dominated phenomena. Application of this method to solving thermoelasticity problems is not so popular, because of disadvantages of DGM associated with such type of equations. The method doesn't handle second order differential equations for steady-state but it is not impossible to solve them using appropriate methodology.

Changes in temperatures cause thermal effects on materials. Some of these effects are thermal stress, strain and deformation. Thermal deformation simply means as the temperature of material increases the material will expand. Of course if the temperature decreases the material will shrink.

## 1. Mathematical description of the problem

Lets consider two-dimensional rigid body  $\Omega$  (Fig. 1) limited by the boundary  $\partial\Omega$ . The body is constrained on the fragment of the boundary because the equilibrium has to be maintained. The temperature of the body is known.

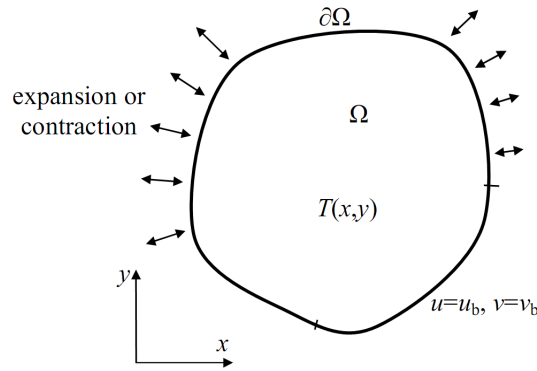


Fig. 1. Thermal effects in the two-dimensional constrained rigid body

Displacement equations of linear thermoelasticity are written in following form

$$\begin{aligned} f_1 \frac{\partial^2 u}{\partial x^2} + f_3 \frac{\partial^2 u}{\partial y^2} + f_3 \frac{\partial^2 v}{\partial y \partial x} + f_2 \frac{\partial^2 v}{\partial x \partial y} &= f_4 \alpha \frac{\partial T}{\partial x} \\ f_2 \frac{\partial^2 u}{\partial y \partial x} + f_3 \frac{\partial^2 u}{\partial x \partial y} + f_3 \frac{\partial^2 v}{\partial x^2} + f_1 \frac{\partial^2 v}{\partial y^2} &= f_4 \alpha \frac{\partial T}{\partial y} \end{aligned} \quad (1)$$

where  $u, v$  [m] - are  $x$  and  $y$  components of the displacement vector respectively,  $T$  [K] is known temperature of the considered body,  $\alpha$  [K<sup>-1</sup>] is linear coefficient of the thermal expansion and  $f_1, f_2, f_3, f_4$  are constants depending on Young modulus  $E$  [N/m<sup>2</sup>] and Poisson's ratio  $\nu$  [-]. Values of these constants differ for the state of plane stress and plane strain in the way showed in the table below.

Table 1

Values of the constants of elasticity for plane stress and plane strain

Constant of elasticity	Plane stress	Plane strain
$f_1$	$\frac{E}{1-\nu^2}$	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$
$f_2$	$\frac{E\nu}{1-\nu^2}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$
$f_3$	$\frac{E}{2(1+\nu)}$	$\frac{E}{2(1+\nu)}$
$f_4$	$f_1 + f_2$	$f_1 + 2f_2$

## 2. Discontinuous formulation

DGM is local, so the considerations are carried out at the level of a single finite element. Rigid body showed in Figure 1 is triangulated, and each triangle from the mesh has its own boundary with unique nodes (Fig. 2). Active element surrounded by its nearest neighbours is only needed to perform calculations. There is no need to aggregate all elements in the huge stiffness matrix and then solve set of thousands equations.

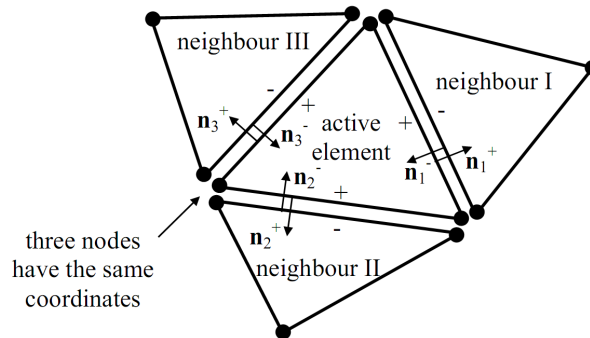


Fig. 2. Active element with its three neighbours form standard DG stencil

DG doesn't handle second order differential terms, thus all of the terms on the left side of (1) have to be rewritten with use of additional variables. One can introduce four new variables (degrees of freedom) into set (1). These variables are in fact spatial derivatives of displacement  $u$  and  $v$ :

$$\begin{aligned} q_x^{(u)} &= \frac{\partial u}{\partial x}, & q_y^{(u)} &= \frac{\partial u}{\partial y} \\ q_x^{(v)} &= \frac{\partial v}{\partial x}, & q_y^{(v)} &= \frac{\partial v}{\partial y} \end{aligned} \quad (2)$$

Substituting new variables into (1) first order differential equations are obtained

$$\begin{aligned} f_1 \frac{\partial q_x^{(u)}}{\partial x} + f_3 \frac{\partial q_y^{(u)}}{\partial y} + f_3 \frac{\partial q_x^{(v)}}{\partial y} + f_2 \frac{\partial q_y^{(v)}}{\partial x} &= f_4 \alpha \frac{\partial T}{\partial x} \\ f_2 \frac{\partial q_x^{(u)}}{\partial y} + f_3 \frac{\partial q_y^{(u)}}{\partial x} + f_3 \frac{\partial q_x^{(v)}}{\partial x} + f_1 \frac{\partial q_y^{(v)}}{\partial y} &= f_4 \alpha \frac{\partial T}{\partial y} \end{aligned} \quad (3)$$

Equations (3)-(4) are supplemented by Dirichlet and Neumann boundary conditions

$$\begin{aligned} (x, y) \in \partial\Omega : u = u_b, v = v_b \\ (x, y) \in \partial\Omega : q_x^{(u)} = q_{xb}^{(u)}, \quad q_y^{(u)} = q_{yb}^{(u)}, \quad q_x^{(v)} = q_{xb}^{(v)}, \quad q_y^{(v)} = q_{yb}^{(v)} \end{aligned} \quad (4)$$

Finally the set of six basic equations is built with six unknowns  $q_x^{(u)}$ ,  $q_y^{(u)}$ ,  $q_x^{(v)}$ ,  $q_y^{(v)}$ ,  $u$ ,  $v$ . This is of course serious drawback of the method, but fortunately four spatial derivatives can be directly used to calculate strain tensor  $\mathbf{T}_\varepsilon$ .

The weighted residual method [4] is used for equations (2)-(3). Multiplying them by the trial function  $w$  and integrating over element  $\Omega^{(e)}$  one can write:

$$\begin{aligned} \int_{\Omega^{(e)}} w q_x^{(u)} dV = \int_{\Omega^{(e)}} w \frac{\partial u}{\partial x} dV, \quad \int_{\Omega^{(e)}} w q_y^{(u)} dV = \int_{\Omega^{(e)}} w \frac{\partial u}{\partial y} dV \\ \int_{\Omega^{(e)}} w q_x^{(v)} dV = \int_{\Omega^{(e)}} w \frac{\partial v}{\partial x} dV, \quad \int_{\Omega^{(e)}} w q_y^{(v)} dV = \int_{\Omega^{(e)}} w \frac{\partial v}{\partial y} dV \end{aligned} \quad (5)$$

$$\begin{aligned} \int_{\Omega^{(e)}} w \left( f_1 \frac{\partial q_x^{(u)}}{\partial x} + f_3 \frac{\partial q_y^{(u)}}{\partial y} + f_3 \frac{\partial q_x^{(v)}}{\partial y} + f_2 \frac{\partial q_y^{(v)}}{\partial x} \right) dV = \int_{\Omega^{(e)}} w f_4 \alpha \frac{\partial T}{\partial x} dV \\ \int_{\Omega^{(e)}} w \left( f_2 \frac{\partial q_x^{(u)}}{\partial y} + f_3 \frac{\partial q_y^{(u)}}{\partial x} + f_3 \frac{\partial q_x^{(v)}}{\partial x} + f_1 \frac{\partial q_y^{(v)}}{\partial y} \right) dV = \int_{\Omega^{(e)}} w f_4 \alpha \frac{\partial T}{\partial y} dV \end{aligned} \quad (6)$$

Exact solution  $q_x^{(u)}$ ,  $q_y^{(u)}$ ,  $q_x^{(v)}$ ,  $q_y^{(v)}$ ,  $u$ ,  $v$  is approximated by functions  $q_{xh}^{(u)}$ ,  $q_{yh}^{(u)}$ ,  $q_{xh}^{(v)}$ ,  $q_{yh}^{(v)}$ ,  $u_h$ ,  $v_h$ . For example the approximation of  $q_x^{(u)}$  and  $u$  is defined as follows:

$$q_{xh}^{(u)} = \sum_{j=1}^N \phi_j q_{xj}^{(u)} \quad (7)$$

$$u_h = \sum_{i=1}^N \phi_i u_i \quad (8)$$

where  $N$  is number of nodes in the element,  $\phi_j$  is arbitrary chosen  $j$ -th approximation function (i.e. shape function).

Inserting approximations (7)-(8) into first equation from (5) and integrating term on its right side by parts one can obtain

$$\int_{\Omega^{(e)}} w q_{xh}^{(u)} dV = - \int_{\Omega^{(e)}} \frac{\partial w}{\partial x} u_h dV + \int_{\partial\Omega^{(e)}} w n_x \hat{u}_h ds \quad (9)$$

Other equations from (5) are written analogously. Equations (6) have to be rearranged in the similar way. Let's do this with respect to the first equation. First derivative term from it has to be integrated by parts

$$\int_{\Omega^{(e)}} w \left( f_1 \frac{\partial q_{xh}^{(u)}}{\partial x} \right) dV = - \int_{\Omega^{(e)}} f_1 \frac{\partial w}{\partial x} q_{xh}^{(u)} dV + \int_{\partial\Omega^{(e)}} f_1 w n_x \hat{q}_{xh}^{(u)} ds \tag{10}$$

where  $n_x, n_y$  are components of vector outward normal to the boundary of active element (Fig. 2),  $\hat{q}_{xh}^{(u)}$  and  $\hat{u}_h$  are numerical fluxes which can be variously defined [5]. In this paper LDG (Local Discontinuous Galerkin) central fluxes were used:

$$\hat{u}_h = \frac{1}{2} (u_h^+ + u_h^-) \tag{11}$$

$$\hat{q}_{xh}^{(u)} = \frac{1}{2} (q_{xh}^{(u)+} + q_{xh}^{(u)-}) - C_{11} (n_x^+ u_h^+ + n_x^- u_h^-) \tag{12}$$

where “+” and “-“ are values taken from active element (+) or its neighbours (-),  $C_{11}$  is the constant affecting convergence ratio of calculation process.

Substituting (11) into (9) following equation is obtained

$$\int_{\Omega^{(e)}} w q_{xh}^{(u)} dV = - \int_{\Omega^{(e)}} \frac{\partial w}{\partial x} u_h dV + \int_{\partial\Omega^{(e)}} w n_x \frac{1}{2} (u_h^+ + u_h^-) ds \tag{13}$$

Finally the equation above is rewritten in following form

$$\int_{\Omega^{(e)}} w q_{xh}^{(u)} dV + \int_{\Omega^{(e)}} \frac{\partial w}{\partial x} u_h dV - \frac{1}{2} n_x \int_{\partial\Omega^{(e)}} w u_h^+ ds = \frac{1}{2} n_x \int_{\partial\Omega^{(e)}} w n_x u_h^- ds \tag{14}$$

The term (10) is rearranged analogously with use of flux (12)

$$\int_{\Omega^{(e)}} w \left( f_1 \frac{\partial q_{xh}^{(u)}}{\partial x} \right) dV = - \int_{\Omega^{(e)}} f_1 \frac{\partial w}{\partial x} q_{xh}^{(u)} dV + \int_{\partial\Omega^{(e)}} f_1 w n_x \left[ \frac{1}{2} (q_{xh}^{(u)+} + q_{xh}^{(u)-}) - C_{11} (n_x^+ u_h^+ + n_x^- u_h^-) \right] ds \tag{15}$$

There are four derivative terms in the first equation from (6). Each of them has to be rewritten in the same way leading to the relation showed below

$$\begin{aligned}
& -f_1 \int_{\Omega^{(e)}} \frac{\partial w}{\partial x} q_{xh}^{(u)} dV - f_3 \int_{\Omega^{(e)}} \frac{\partial w}{\partial y} q_{yh}^{(u)} dV - f_3 \int_{\Omega^{(e)}} \frac{\partial w}{\partial y} q_{xh}^{(v)} dV - f_2 \int_{\Omega^{(e)}} \frac{\partial w}{\partial x} q_{yh}^{(v)} dV + \\
& + \frac{1}{2} \left[ n_x \left( f_1 \int_{\partial\Omega^{(e)}} w q_{xh}^{(u)+} ds + f_2 \int_{\partial\Omega^{(e)}} w q_{yh}^{(v)+} ds \right) + f_3 n_y \left( \int_{\partial\Omega^{(e)}} w q_{yh}^{(u)+} ds + \int_{\partial\Omega^{(e)}} w q_{xh}^{(v)+} ds \right) \right] + \\
& - C_{11} (f_1 n_x^2 + f_3 n_y^2) \int_{\partial\Omega^{(e)}} w u_h^+ ds - C_{11} n_x n_y (f_2 + f_3) \int_{\partial\Omega^{(e)}} w u_h^+ ds = -f_4 \alpha \int_{\Omega^{(e)}} \frac{\partial w}{\partial x} T_h dV + \quad (16) \\
& - \frac{1}{2} \left[ n_x \left( f_1 \int_{\partial\Omega^{(e)}} w q_{xh}^{(u)-} ds + f_2 \int_{\partial\Omega^{(e)}} w q_{yh}^{(v)-} ds \right) + f_3 n_y \left( \int_{\partial\Omega^{(e)}} w q_{yh}^{(u)-} ds + \int_{\partial\Omega^{(e)}} w q_{xh}^{(v)-} ds \right) \right] + \\
& - C_{11} (f_1 n_x^2 + f_3 n_y^2) \int_{\partial\Omega^{(e)}} w u_h^- ds - C_{11} n_x n_y (f_2 + f_3) \int_{\partial\Omega^{(e)}} w v_h^- ds + f_4 \alpha \int_{\partial\Omega^{(e)}} w n_x T_h ds
\end{aligned}$$

Finally the set of six equations is built. Four of them look like (14), two of them have the form of (16). Trial function  $w$  is taken as shape function  $\phi$ , thus for  $N$  nodes in the element one can write

$$\mathbf{\Phi} = (\phi_1, \phi_2, \dots, \phi_N)^T \quad (17)$$

Integral terms are showed in matrix form

$$\begin{aligned}
& \begin{bmatrix} \mathbf{M} & 0 & 0 & 0 & \mathbf{D}_x & 0 \\ 0 & \mathbf{M} & 0 & 0 & \mathbf{D}_y & 0 \\ 0 & 0 & \mathbf{M} & 0 & 0 & \mathbf{D}_x \\ 0 & 0 & 0 & \mathbf{M} & 0 & \mathbf{D}_y \\ -f_1 \mathbf{D}_x & -f_3 \mathbf{D}_y & -f_3 \mathbf{D}_y & -f_2 \mathbf{D}_x & -\mathbf{S}_x^{(1)} & -\mathbf{S}_y \\ -f_2 \mathbf{D}_y & -f_3 \mathbf{D}_x & -f_3 \mathbf{D}_x & -f_1 \mathbf{D}_y & -\mathbf{S}_y & -\mathbf{S}_x^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_x^{(u)} \\ \mathbf{q}_y^{(u)} \\ \mathbf{q}_x^{(v)} \\ \mathbf{q}_y^{(v)} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \\
& = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{B}_x & 0 \\ 0 & 0 & 0 & 0 & \mathbf{B}_y & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{B}_x \\ 0 & 0 & 0 & 0 & 0 & \mathbf{B}_y \\ -f_1 \mathbf{B}_x & -f_3 \mathbf{B}_y & -f_3 \mathbf{B}_y & -f_2 \mathbf{B}_x & -\mathbf{S}_x^{(1)} & -\mathbf{S}_y \\ -f_2 \mathbf{B}_y & -f_3 \mathbf{B}_x & -f_3 \mathbf{B}_x & -f_1 \mathbf{B}_y & -\mathbf{S}_y & -\mathbf{S}_x^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_x^{(u)-} \\ \mathbf{q}_y^{(u)-} \\ \mathbf{q}_x^{(v)-} \\ \mathbf{q}_y^{(v)-} \\ \mathbf{u}^- \\ \mathbf{v}^- \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathbf{H}_x \\ \mathbf{H}_y \end{bmatrix} \quad (18)
\end{aligned}$$

where successive matrices are in fact integrals calculated as follows:

$$\mathbf{M} = \int_{\Omega^{(e)}} \mathbf{\Phi} \mathbf{\Phi}^T dV \quad (19)$$

$$\mathbf{D}_x = \int_{\Omega^{(e)}} \left( \frac{\partial \Phi}{\partial x} \right) \Phi^T dV - \mathbf{B}_x, \quad \mathbf{B}_x = \frac{1}{2} n_x \int_{\partial \Omega^{(e)}} \Phi \Phi^T ds \quad (20)$$

$$\mathbf{D}_y = \int_{\Omega^{(e)}} \left( \frac{\partial \Phi}{\partial y} \right) \Phi^T dV - \mathbf{B}_y, \quad \mathbf{B}_y = \frac{1}{2} n_y \int_{\partial \Omega^{(e)}} \Phi \Phi^T ds \quad (21)$$

$$\mathbf{S}_x^{(1)} = C_{11} (f_1 n_x^2 + f_3 n_y^2) \int_{\partial \Omega^{(e)}} \Phi \Phi^T ds \quad (22)$$

$$\mathbf{S}_x^{(2)} = C_{11} (f_3 n_x^2 + f_1 n_y^2) \int_{\partial \Omega^{(e)}} \Phi \Phi^T ds \quad (23)$$

$$\mathbf{S}_y = C_{11} (f_2 + f_3) n_x n_y \int_{\partial \Omega^{(e)}} \Phi \Phi^T ds \quad (24)$$

$$\mathbf{H}_x = -f_4 \alpha \int_{\Omega^{(e)}} \left( \frac{\partial \Phi}{\partial x} \right) \Phi^T \mathbf{T} dV + f_4 \alpha \int_{\partial \Omega^{(e)}} \Phi \Phi^T \mathbf{T} ds \quad (25)$$

$$\mathbf{H}_y = -f_4 \alpha \int_{\Omega^{(e)}} \left( \frac{\partial \Phi}{\partial y} \right) \Phi^T \mathbf{T} dV + f_4 \alpha \int_{\partial \Omega^{(e)}} \Phi \Phi^T \mathbf{T} ds \quad (26)$$

The matrices (18) can be combined to yield the following resultant matrix

$$\mathbf{K}\mathbf{U} = \mathbf{F} \quad (27)$$

where  $\mathbf{K}$  is the stiffness matrix,  $\mathbf{U}$  is the vector contains unknowns and  $\mathbf{F}$  is the right hand side vector.

The computational procedure is iterative element-by-element solution. Boundary conditions are gradually propagated into the domain during this process. The calculations start with an element located at the boundary and progressively sweep through the region. This is repeated until difference between results from previous and current iteration is smaller than  $\varepsilon$  (i.e.  $\varepsilon = 10^{-6}$ ). Obtained  $q_x^{(u)}$ ,  $q_y^{(u)}$ ,  $q_x^{(v)}$ ,  $q_y^{(v)}$  can be used to calculate components of the strain tensor. Further they can be used to compute components of the stress tensor.

## Conclusions

Presented mathematical and discontinuous numerical model of the planar thermoelasticity problem shows methodology of Discontinuous Galerkin Method for

solving second order partial differential equations. On the basis of these considerations a computer program can be easily constructed.

### References

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