# APPLICATION OF HM-NETWORKS IN PROBLEMS OF TRANSPORT LOGISTICS

*Michal Matalycki[1], Olga Kiturko[2], Natalia Basiewicz[2]*

[1]*Institute of Mathematics and Computer Science, Czestochowa University of Technology Poland*
[2]*Faculty of Mathematics and Computer Science, Grodno State University, Belarus*
*sytaya_om@mail.ru*

**Abstract.** The article deals with the technique allowing to estimate and forecast expected incomes, logistics transport systems subjects warehouse squares. The technique is based on application of HM (Howard-Matalytski) - queueing networks with incomes.

## Introduction

The logistics structure is a system which consists of various functional areas, such as: stores, information, stacking and warehouse handling, transporting of products and other areas [1]. Thus the main task is cost minimization and profit maximization for producers', customers', warehouses' etc.

Logistics systems (LS) differ in their structure, enterprise sizes, functions, a warehouse economy, transport models and so on. Structural modifications in a carrier render, prices for fuel and other material resources, scientific and technical process influence them a lot. The latter causes transport services cost modification. As a result, transport strategy and the whole LS is overestimated.

Working out LS models is an important problem. The number and the arrangement of producing units (the enterprises, firms, etc.), an amount and the arrangement of warehouses, transport models, connection and information systems are to be taken into consideration.

According to the analysis routing is logistic's basics in the sphere of transportations. There are three areas in this sphere: perfection of available algorithms, working out new economic-mathematical models which would reflect promoting of material stream better, confluence of routing models with other functional logistic models, such as storekeeping.

The problems, being solved by LS, can basically be divided into three groups:

1. The problems connected with market service zones shaping, material stream prognosis, material stream handling in serving system (a warehouse of a supplier/customer, an enterprise of wholesale trade).
2. The problems including transportation organizing system engineering (transportations plans, activity distribution plans, goods traffic planning, train diagram of vehicles, etc.).

3. The problems connected with storekeeping at enterprises, firms, warehouse complexes, arrangement of stores and servicing them by vehicles and information systems.

These problems' solution is based on working out the strategy and logistics concept of transport service model for customers and firms which is based on efficient transportation routes and schedule compiling for production delivery to customers, i.e. routing of transportations.

The efficiency of LS does not only depend on perfect and intensive industrial and transport manufacturing, but on warehousing as well. Warehousing contributes to quality preservation of products, materials and raw materials; it increases steady and organized manufacturing and transport work; refines the use of enterprises' territories; decreases transport idle standing and transportation cost; it prevents workers from inefficient cargo handling and extra warehouse labour to use their efforts in mainline manufacturing.

Warehouses of various types can be formed in the beginning, middle and in the end of goods traffic or manufacturing for temporary cargo accumulation and opportune supply of manufacture with necessary amount of materials. They shouldn't be considered only as devices for cargo storage, but as transport warehouse complexes as well in which cargo transition processes play an important role. The work of these complexes has dynamic, stochastic character due to non-uniformity of cargo transportations.

Usually economic, economic-mathematical, statistical methods are used to solve theoretical and practical logistic problems. Streams of production, arriving from producers and addressees at stochastic time moments and time intervals which are necessary for manufacturing, product unloading, staying in warehouses and sales have predetermined the necessity of queueing theory methods use for working out mathematical models applied in logistics. In [1] the application of Markov's queueing systems (QS) when defining the number of transport warehouse workers in a brigade for cargo loading according to the technological scheme: a warehouse – a loader - the car is described. It's clear that in case we want to describe functioning of various producers, warehouses and customers of the LS as a united system, queueing network (QN) consisting of various QS, corresponding to the subjects of the LS can serve as the model of such functioning. The important problems for logistics transport systems (LTS) are problems of estimation and forecasting of their subjects' incomes, received from product transportation between subjects by various means of transport. Let's assume that LTS consists of $n$ subjects $S_1$, $S_2, \ldots, S_n$ (factories-manufacturers, warehouses, addresses) who carry out cargo transportation. Transportation by car brings some income, received by realization of product, and the income of one subject decreases that of the other one. Thus transport expenses (fuel, car repairs), a driver's salary can be referred to either of the subjects. The «unloading - loading» car time within LTS subjects and car streams between them are casual. It is necessary to estimate (to predict) expected (average) incomes of LTS subjects carrying out such transportations. A similar situation arises with estimation and forecasting of transport agency (TA) incomes from cargo transportation between subjects. Cargo transportation from the subject $S_i$ to the subject $S_j$ brings TA some casual income, however TA incurs some losses. To solve these problems a new class of queueing networks - Markov networks with the incomes, recently named HM (Howard-Matalytski) networks [3, 4] which can be used to forecast incomes of various systems, not necessarily logistic [5] is being introduced.

## 1. HM-networks application to LTS subjects' expected incomes forecasting

Let's designate by $k(t)=(k,t)=(k_1(t),k_2(t),...,k_n(t))$ - LTS condition vector, where $k_i(t)$ - a number of cars in point $S_i$ (being in turn and at unloading-loading) in the moment $t$. From the outside of LTS the elementary stream of cars with intensity $\lambda(t)$ arrives. We will consider that cars «unloading - loading» time intervals in $S_i$ are distributed under the exponential law with parameter $\mu_i(k_i(t))$, $i=\overline{1,n}$. It means that our exponential network is in $(k,t)$ condition, where $k_i(t)$ - number of requests in *i*-th QS, $\mu_i(k_i(t))$ - the requests' service intensity in *i*-th QS, $i=\overline{1,n}$.

Let's consider a case when transition incomes between network conditions are determined by functions that depend on condition of the network and time, and QS networks are one-linear. Let $v_i(k,t)$ - the full expected income which is received by system $S_i$ for time $t$, if in an initial instant the network is in the state $k$, and we will assume that this function is differentiated on $t$; $r_i(k)$ - the system income $S_i$ in the unit of time when the network is at the state $k$; $r_{0i}(k+I_i,t)$ - the system income $S_i$, when the network makes transition from the state $(k,t)$ to the state $(k+I_i,t+\Delta t)$ for time $\Delta t$; $-R_{i0}(k-I_i,t)$ - the income of this system if the network makes transition from the state $(k,t)$ to the state $(k-I_i,t+\Delta t)$; $r_{ij}(k+I_i-I_j,t)$ - the system income $S_i$ (the expense or a system loss $S_j$) when the network changes the state from $(k,t)$ on $(k+I_i-I_j,t+\Delta t)$ for time $\Delta t$, $i,j=\overline{1,n}$.

Let the network be in the state $(k,t)$. During the interval of time $\Delta t$ it can remain in this state or pass to states $(k-I_i,t+\Delta t)$, $(k+I_i,t+\Delta t)$, $(k+I_i-I_j,t+\Delta t)$, $i,j=\overline{1,n}$. If the network remains in the state $(k,t+\Delta t)$ expected income of system $S_i$ will make a $r_i(k)\Delta t$ plus the expected income $v_i(k,t)$, which it will receive for remained $t$ time units. The probability of such event is equal $1-\left(\lambda(t)+\sum_{j=1}^{n}\mu_j(k_j(t))u(k_j(t))\right)\Delta t+o(\Delta t)$. If the network transits to the state $(k+I_i,t+\Delta t)$ with probability $\lambda(t)p_{0i}\Delta t+o(\Delta t)$ system income $S_i$ will make $[r_{0i}(k+I_i,t)+v_i(k+I_i,t)]$, and if to the state $(k-I_i,t+\Delta t)$ with probability the $\mu_i(k_i(t))u(k_i(t))p_{i0}\Delta t+o(\Delta t)$ income of this system will make $[-R_{i0}(k-I_i,t)+v_i(k-I_i,t)]$, $i=\overline{1,n}$. Similarly, if the network transits from the state $(k,t)$ to the state $(k+I_i-I_j,t+\Delta t)$ with probability $\mu_j(k_j(t))u(k_j(t))p_{ji}\Delta t+o(\Delta t)$, it brings into system the $S_i$ income in a size of $r_{ij}(k+I_i-I_j,t)$ plus the expected income of a network for remained time if the state was an initial state of the network $(k+I_i-I_j)$. We will tabulate mentioned above (table 1).

Then, using the formula of total probability for expectation, for the expected income of system $S_i$ it is possible to receive the system of difference-differential equations (DDE):

$$\frac{dv_i(k,t)}{dt}=-\left[\lambda(t)+\sum_{j=1}^{n}\mu_j(k_j(t))u(k_j(t))\right]v_i(k,t)+$$

$$+\sum_{j=1}^{n}\left[\lambda(t)p_{0j}v_i(k+I_j,t)+\mu_j(k_j(t))u(k_j(t))p_{j0}v_i(k-I_j,t)\right]+$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mu_j(k_j(t))u(k_j(t))p_{ji}v_i(k+I_i-I_j,t)+\mu_i(k_i(t))u(k_i(t))p_{ij}v_i(k-I_i+I_j,t)\right]+$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mu_j(k_j(t))u(k_j(t))p_{ji}r_{ij}(k+I_i-I_j,t)-\mu_i(k_i(t))u(k_i(t))p_{ij}r_{ji}(k-I_i+I_j,t)\right]+$$

$$+\sum_{\substack{c,s=1\\ c,s\neq i}}^{n}\mu_s(k_s(t))p_{sc}v_i(k+I_c-I_s,t)+\lambda(t)p_{0i}r_{0i}(k+I_i,t)-$$

$$-\mu_i(k_i(t))u(k_i(t))p_{i0}R_{i0}(k-I_i,t)+r_i(k) \qquad (1)$$

The number of the equations in this system is equal to the number of network states.

Table 1

**Possible transitions between states of the network, their probabilities and system incomes $S_i$**

| Possible transitions between network states | Probabilities of transitions | Incomes of system $S_i$ from transitions between their states |
|---|---|---|
| $(k,t) \to (k, t+\Delta t)$ | $1-\big(\lambda(t) + \sum_{j=1}^{n} \mu_j(k_j(t))u(k_j(t))\big)\Delta t + o(\Delta t)$ | $r_i(k)\Delta t + v_i(k,t)$ |
| $(k,t) \to (k+I_j, t+\Delta t)$, $j \ne i$ | $\lambda(t)p_{0j}\Delta t + o(\Delta t)$ | $r_i(k)\Delta t + v_i(k+I_j,t)$ |
| $(k,t) \to (k-I_j, t+\Delta t)$, $j \ne i$ | $\mu_j(k_j(t))u(k_j(t))p_{j0}\Delta t + o(\Delta t)$ | $r_i(k)\Delta t + v_i(k-I_j,t)$ |
| $(k,t) \to (k+I_c-I_s, t+\Delta t)$, $c,s \ne i$ | $\mu_s(k_s(t))u(k_s(t))p_{sc}\Delta t + o(\Delta t)$ | $r_i(k)\Delta t + v_i(k+I_c-I_s,t)$ |
| $(k,t) \to (k+I_i, t+\Delta t)$ | $\lambda(t)p_{0i}\Delta t + o(\Delta t)$ | $r_{0i}(k+I_i,t) + v_i(k+I_i,t)$ |
| $(k,t) \to (k-I_i, t+\Delta t)$ | $\mu_i(k_i(t))u(k_i(t))p_{i0}\Delta t + o(\Delta t)$ | $-R_{i0}(k-I_i,t) + +v_i(k-I_i,t)$ |
| $(k,t) \to (k+I_i-I_j, t+\Delta t)$, $j \ne i$ | $\mu_j(k_j(t))u(k_j(t))p_{ji}\Delta t + o(\Delta t)$ | $r_{ij}(k+I_i-I_j,t) + v_i(k+I_i-I_j,t)$ |
| $(k,t) \to (k-I_i+I_j, t+\Delta t)$, $j \ne i$ | $\mu_i(k_i(t))u(k_i(t))p_{ij}\Delta t + o(\Delta t)$ | $-r_{ji}(k-I_i+I_j,t) + v_i(k-I_i+I_j,t)$ |

Let's consider the case when intensity $\lambda(t)=\lambda$, $\mu_i(k_i(t)) = \mu_i(k_i)$, $i=\overline{1,n}$, does not depend on time. In this case for the closed networks the set of equations (1) can come to a system of a finite number linear inhomogeneous UDE with constant factors which in the matrix form can be noted in an aspect

$$\frac{dV_i(t)}{dt} = Q_i(t) + AV_i(t) \tag{2}$$

where: $V_i^T(t) = \big(v_i(1,t), v_i(2,t), ..., v_i(l,t)\big)$ - a required vector of system incomes $S_i$, $l$ - number of network states. The solution of the system (2) can be discovered, using a direct method or a method of Laplace transformations.

Let's consider both methods in a more detailed way. For the solution of system (2) by direct method having multiplied both parts of the system (2) by $e^{-At}$, we will receive

$$e^{-At}V'(t) = e^{-At}AV(t) + e^{-At}Q_i(t)$$

or

$$e^{-At}\left(-AV(t)+V'(t)\right)=\frac{d}{dt}\left(e^{-At}V(t)\right)=e^{-At}Q_i(t)$$

so

$$e^{-At}V(t)=V(0)+\int_0^t e^{-A\tau}Q_i(\tau)d\tau$$

that is

$$V(t)=e^{At}V(0)+\int_0^t e^{A(t-\tau)}Q_i(\tau)d\tau \tag{3}$$

where $e^{At}=I+At+\frac{A^2t^2}{2!}+\ldots+\frac{A^mt^m}{m!}+\ldots$ - a matrix exponential curve, $I$ - an identity matrix. For determination of the matrix $e^{At}$ we should discover eigenvalues $q_1,q_2,\ldots,q_l$ of a matrix $A$ and a complete set of right eigenvectors corresponding to them $u^{(1)},u^{(2)},\ldots,u^{(l)}$ if it is possible. Then we should use representation

$$e^{At}=UB(t)U^{-1} \tag{4}$$

where $U$ - a matrix the columns of which are eigenvectors $u^{(1)},u^{(2)},\ldots,u^{(l)}$, $B(t)$ - a scalar matrix

$$B(t)=\begin{pmatrix} e^{q_1t} & 0 & \cdots & 0 \\ 0 & e^{q_2t} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{q_lt} \end{pmatrix} \tag{5}$$

For the solution of system (2) by the method of Laplace transformations, we will set a vector of entry states $V_i(0)$. Let $U_i(s)$ be a vector of Laplace transformations of incomes $v_i(j,t)$, $i=\overline{1,l}$, $G_i(s)$ - a Laplace transformation $Q_i(t)$. Then $sU_i(s)-V_i(0)=G_i(s)+AU_i(s)$ or $(sI-A)U_i(s)=G_i(s)+V_i(0)$. From here we discover $U_i(s)$:

$$U_i(s)=(sI-A)^{-1}G_i(s)+(sI-A)^{-1}V_i(0) \tag{6}$$

The vector of incomes $V_i(t)$ can be discovered by means of inverse transformation for (6). If $W(t)$ - an inverse Laplace transformation of a matrix $(sI-A)^{-1}$ correspondingly inverse transformation translates a relation (6) in (7):

$$V_i(t) = W(t) * Q_i(t) + W(t)\,V_i(0)\,,\ i = \overline{1,n} \qquad (7)$$

where $W(t) * Q_i(t) = \int_0^t W(u) Q_i(t-u) du$. In case when incomes from transitions between network states are determined magnitudes depending on states of the network, the set of equations (1) will look as $\frac{dV_i(t)}{dt} = Q_i + AV_i(t)$, and its solution $V_i(t) = W(t)(Q_i + V_i(0))$, $i = \overline{1,n}$. However, we shouldn't forget that the number of closed QN states is equal to $l = C_{n+K-1}^{n-1}$, where $K$ - the number of the requests served in the network, and it is big enough considering rather small $n$ and $K$, i.e. the number of the equations in system (2) will also be big enough. Experience has shown that such methods are possible to use when make calculations for networks with rather small state space ($l$ < 100); direct method can be used for bigger dimension networks than the method of Laplace transformations.

## 2. Analysis of the goods transporting model

Let's consider closed Markov HM-network with the same type of requests, consisting of $M = n + m_1 + ... + m_{n-1}$ service systems $S_i$, $i = 1,..n, 1_1,...,1_{m_1},...,(n-1)_1,...,(n-1)_{m_{(n-1)}}$, represented by Figure 1, which illustrates a model of goods transporting. In this model the system $S_n$ is "transport agency" (TA), and under systems $S_1$, $S_2, \ldots, S_{n-1}$ we will understand «warehouses of consignors» where some goods are stored; $S_{i_1}$, $S_{i_2}, \ldots, S_{i_{m_i}}$ - «consignees» (their warehouses, sale points of goods which have arrived from the warehouse $S_i$, $i = \overline{1,(n-1)}$). Depending on the vehicle (V) chosen there can be various expenses, and consequently transitions from TA $S_n$ to a particular "the warehouse of the consignor" $S_1$, $S_2, \ldots, S_{n-1}$. Thus the request is understood as the enterprise V transition from one system to another with the purpose of goods transportation in logistics system «transport agency - the warehouse of the consignor - consignees».

Let's introduce some sets: $X_i = \{i_1, i_2, ..., i_{m_i}\}$, $i = \overline{1,(n-1)}$; $X_0 = \bigcup_{i=1}^{n-1} X_i$; $X = \{1,2,...,n\} \bigcup X_0$. As a network state in the moment $t$ we will understand a vector $(k,t) = (k_1, k_2, \ldots, k_n, ..., k_{1_1}, ..., k_{1_{m_1}}, ..., k_{(n-1)_1}, ..., k_{(n-1)_{m_{(n-1)}}}, t)$, where $k_i$ is a number of requests in system $S_i$ in the moment $t$, $i \in X$. The number of network states is equal $l = C_{M+K-1}^{M-1}$, where $K$ is number of requests in the network.

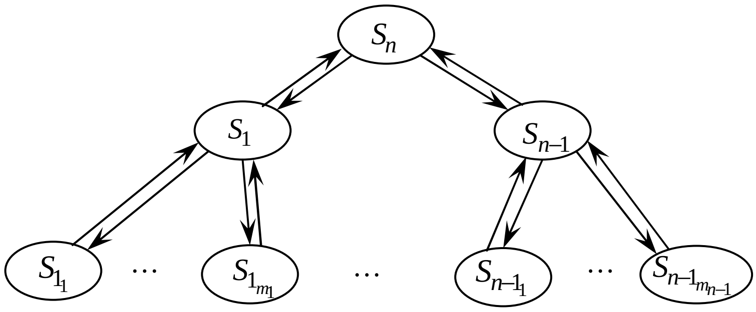


Fig. 1. Network model of goods transporting

Let $v_i(k,t)$ be the full expected income which is received by system $S_i$, for time $t$, if in at the initial instant the network is in the state $k$; $r_{ij}(k,t)$ is the system $S_i$ income, and also system $S_j$ loss or expense respectively in time $\Delta t$ if the network makes transition to state $(k,t+\Delta t)$ during this time; $r_i(k)$ - the system $S_i$ income in c.u. during the unit of time when network is in the state $k$, $i, j \in X$.

Let's consider a case when parameters of request service in QS networks depend on time only, i.e. if the request is being served in *i*-th QS in the moment $t$ it will be served in time interval $[t, t+\Delta t)$ with probability $\mu_i(t)\Delta t + o(\Delta t),\ i \in X$.

Let the HM-network be in the state $(k,t)$. We will assume that the system $S_n$ receives the income of $r_n(k)$ c.u. for the time unit during the phase when the network is in the state $k$. If it remains in the state $(k,t)$ during the time interval $\Delta t$ the expected income of system $S_n$ will make $r_n(k)\Delta t$ plus the expected income $v_n(k,t)$ which it will bring for remained $t$ time units. The probability of such event is equal to $1-\sum_{j\in X}\mu_j(t)u(k_j)\Delta t + o(\Delta t)$, where $u(x)=\begin{cases}1, & x>0,\\ 0, & x\le 0,\end{cases}$ is the function of Heaviside. When the network makes transition from the state $(k,t)$ to the state $(k-I_j+I_n, t+\Delta t)$ during time $\Delta t$ with probability $\mu_j(t)u(k_j)p_{jn}\Delta t + o(\Delta t)$, it yields a loss to the system $S_n$ at the size of $r_{jn}(k-I_j+I_n,t)$, and the expected income of system $S_n$ will make $-r_{jn}(k-I_j+I_n,t)$ plus the expected income $v_n(k-I_j+I_n,t)$ which will be received for remained time if an initial network state was $(k-I_j+I_n)$, $j=\overline{1,n-1}$. At the transition described above the V returns to the company territory, and it means that the system $S_n$ suffers losses. Similarly, when the network makes transition from the state $(k,t)$ to state $(k+I_j-I_n, t+\Delta t)$ with probability

$\mu_n(t)u(k_n)p_{nj}\Delta t + o(\Delta t)$, it brings income to the system $S_n$ as large as $r_{nj}(k - I_n + I_j, t)$, that is a vehicle transits to the system «warehouse of the consignor», and, hence, the system $S_n$ receives the income paid by the given warehouse. The expected income of system «TA» will make $r_{nj}(k - I_n + I_j, t)$ plus the expected income of the network for remained time if the initial state of the network was $(k + I_j - I_n)$, $j = \overline{1, n-1}$. When the network makes transition from state $(k, t)$ to state $(k + I_c - I_s, t + \Delta t)$ in time $\Delta t$ with the probability $\mu_s(t)u(k_s)p_{sc}\Delta t + o(\Delta t)$, the expected income of the system $S_n$ will make $r_n(k)\Delta t$ in time $\Delta t$ plus the expected income $v_n(k + I_c - I_s, t)$ which will be received by it for remained time $t$ if an initial state of the network was $(k + I_c - I_s)$, $s = \overline{1, n-1}$, $c = s_1, s_2, ..., s_{m_s}$.

Then for the expected income of system $S_n$ it is possible to note type (1) system DDE, [6]:

$$\frac{dv_n(k,t)}{dt} = r_n(k) - v_n(k,t)\sum_{j \in X} \mu_j(t)u(k_j) +$$

$$\sum_{j=1}^{n-1} \mu_n(t)u(k_n)p_{nj}\left(r_{nj}(k - I_n + I_j, t) + v_n(k - I_n + I_j, t)\right) +$$

$$+ \sum_{j=1}^{n-1} \mu_j(t)u(k_j)p_{jn}\left(v_n(k + I_n - I_j, t) - r_{jn}(k + I_n - I_j, t)\right) +$$

$$+ \sum_{\substack{s=\overline{1,n-1} \\ c=s_1,s_2,\ldots,s_{m_s}}} \mu_s(t)u(k_s)p_{sc}v_n(k - I_s + I_c, t) +$$

$$+ \sum_{\substack{s=\overline{1,n-1} \\ c=s_1,s_2,\ldots,s_{m_s}}} \mu_c(t)u(k_c)v_n(k + I_s - I_c, t) \tag{8}$$

The system (8), in turn, can be reduced to the system of a finite number of inhomogeneous UDE.

Let's rename network states sequentially $1, 2, \ldots, l$. Take a note that it is possible to present a set of equations (8) in a matrix aspect:

$$\frac{dV_n(t)}{dt} = Q_n(t) + A(t)V_n(t) \tag{9}$$

It is possible to solve the set of equations as in (9) by the means of Mathematica 5.1 package.

**Example 1.** Let's consider TA functioning within our LTS. The principal kind of activity is automobile truck transport activity. Several big cars are available at the enterprise. The majority of transported cargoes is foodstuff, perishable products and the goods that require special temperature conditions for transportation. TA renders its services to dozens of various organizations.

Having analyzed the statistics of the enterprise, formed on the basis of travelling sheets data, amount of customers and requests for tours in 2009, the following service intensities presented in table 2 have been discovered. Intensity of service is understood as an amount of tours performed in a unit of time (in a month).

Table 2

**The data on fulfilled flights**

| Phase | Intensity upkeeps |
|---|---|
| January | 8 |
| February | 7 |
| March | 10 |
| April | 8 |
| May | 6 |
| June | 5 |
| July | 4 |
| August | 4 |
| September | 6 |
| October | 13 |
| November | 7 |
| December | 5 |

Using software product Advanced Grapher 2.2 [7] and TA statistics presented in Table 2, the best approximating function for requests service intensity, passing through the points specified in Table 2 has been discovered:

$$\mu_n(t) = 3.671 \cdot 10^{-4} t^8 - 0.019 t^7 + 0.397 t^6 - 4.55 t^5 + 30.534 t^4 -$$

$$- 121.47 t^3 + 274.763 t^2 - 314.882 t + 143.25 \tag{10}$$

Let's describe how it is possible to use the technique described above at forecasting of incomes TA. For example, let's consider that $M = 3$, the scheme of transportations is specified in Figure 3. And let's consider that $K = 4$, $\mu_1 = 5.0$, $\mu_{1_1} = 1.5$, $\mu_2(t)$ looks like in (10).

The system «warehouse of a consignor» we understand as a warehouse of some consignor (in fig. 2 it is system $S_1$), and «consignees» we see as a warehouse of some consignee and other enterprises (on Fig. 2 it is system $S_{1_1}$), the system $S_2$ is TA.

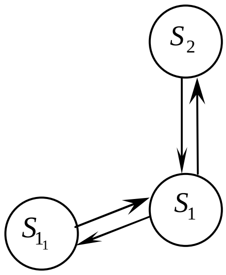


Fig. 2. The scheme of goods transporting at 3 service systems

Let's assume that transitions incomes between states do not depend on states and time. Probabilities matrixes of request transition between QS networks and one-step incomes at requests transition between QS are respectively represented as follows:

$$P = \|p_{ij}\|_{3\times3} = \begin{pmatrix} 0 & 0.4 & 0.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ R = \|r_{ij}\|_{3\times3} = \begin{pmatrix} 3 & 1.2 & 3.2 \\ 1.8 & 5.1 & 0 \\ 0 & 2.5 & 1.9 \end{pmatrix}$$

As the network is closed, the number of its states equals to $l = C_{3+4-1}^{3-1} = 15$.

Let's set a vector of initial states: $V(0) = (0,0,...,0)$. The vector of states looks like $(k,t) = (k_1, k_2, K - k_1 - k_2)$. The states of such network are: $(4,0,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$, $(1,3,0)$, $(1,2,1)$, $(1,1,2)$, $(1,0,3)$, $(0,4,0)$, $(0,3,1)$, $(0,2,2)$, $(0,1,3)$, $(0,0,4)$. We will rename them respectively 1, …, 15. In this case the set of equations (8) looks like:

$$\frac{dv_2(1,t)}{dt} = -5v_2(1,t) + 2v_2(2,t) + 3v_2(3,t) + 2.7,$$

$$\frac{dv_2(2,t)}{dt} = \mu_2(t)v_2(1,t) - (5 + \mu_2(t))v_2(2,t) + 2v_2(4,t) + 3v_2(5,t) + 1.8\mu_2(t) + 2.7,$$

$$\frac{dv_2(3,t)}{dt} = 1.5v_2(1,t) - 6.5v_2(3,t) + 2v_2(5,t) + 3v_2(6,t) + 2.7,$$

$$\frac{dv_2(4,t)}{dt} = \mu_2(t)v_2(2,t) - (5 + \mu_2(t))v_2(4,t) + 2v_2(7,t) + 3v_2(8,t) + 1.8\mu_2(t) + 2.7,$$

$$\frac{dv_2(5,t)}{dt} = 1.5v_2(2,t) + \mu_2(t)v_2(3,t) - (6.5 + \mu_2(t))v_2(5,t) +$$
$$+ 2v_2(8,t) + 3v_2(9,t) + 1.8\mu_2(t) + 2.7,$$

$$\frac{dv_2(6,t)}{dt} = 1.5v_2(3,t) - 6.5v_2(6,t) + 2v_2(9,t) + 3v_2(10,t) + 2.7,$$

$$\frac{dv_2(7,t)}{dt} = \mu_2(t)v_2(4,t) - (5 + \mu_2(t))v_2(7,t) + 2v_2(11,t) + \\ + 3v_2(12,t) + 1.8\mu_2(t) + 2.7,$$

$$\frac{dv_2(8,t)}{dt} = 1.5v_2(4,t) + \mu_2(t)v_2(5,t) - (6.5 + \mu_2(t))v_2(8,t) + 2v_2(12,t) + \\ + 3v_2(13,t) + 1.8\mu_2(t) + 2.7,$$

$$\frac{dv_2(9,t)}{dt} = 1.5v_2(5,t) + \mu_2(t)v_2(6,t) - (6.5 + \mu_2(t))v_2(9,t) + 2v_2(13,t) + \\ + 3 \cdot v_2(14,t) + 1.8\mu_2(t) + 2.7,$$

$$\frac{dv_2(10,t)}{dt} = 1.5v_2(6,t) - 6.5v_2(10,t) + 2v_2(14,t) + 3v_2(15,t) + 2.7,$$

$$\frac{dv_2(11,t)}{dt} = \mu_2(t)v_2(7,t) - \mu_2(t)v_2(11,t) + 1.8\mu_2(t) + 5.1,$$

$$\frac{dv_2(12,t)}{dt} = 1.5v_2(7,t) + \mu_2(t)v_2(8,t) - (\mu_2(t) + 1.5)v_2(12,t) + 1.8\mu_2(t) + 5.1,$$

$$\frac{dv_2(13,t)}{dt} = 1.5v_2(8,t) + \mu_2(t)v_2(9,t) - (\mu_2(t) + 1.5)v_2(13,t) + 1.8\mu_2(t) + 5.1,$$

$$\frac{dv_2(14,t)}{dt} = 1.5v_2(9,t) + \mu_2(t)v_2(10,t) - (\mu_2(t) + 1.5)v_2(14,t) + 1.8\mu_2(t) + 5.1,$$

$$\frac{dv_2(15,t)}{dt} = 1.5v_2(10,t) - 1.5v_2(15,t) + 5.1.$$

In Figure 3 the income modification of "TA" system $v_2(k,t)$ depending on time is presented at the initial networks state $k = (2,0,2)$ discovered by the means of package Mathematica 5.1 and firmware function DSolve [].

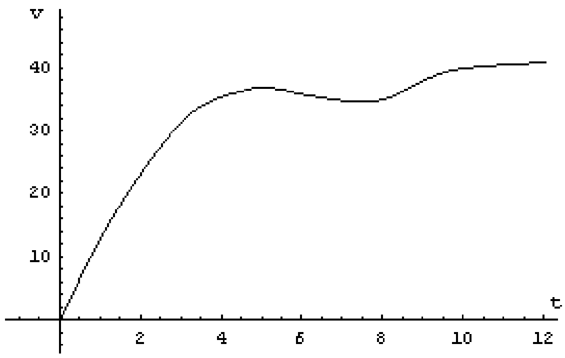


Fig. 3. Incomes modification of "TA" system $v_2(1,2,1,t)$

## 3. Application of HM-networks at warehouse designing

We'd like to mention that the HM-network can be used not only for LTS subjects expected incomes forecasting, but also for designing of warehouse squares, determination of transport warehouse workers amount in brigades that are engaged in cargo loading and unloading. We will illustrate it in the following example. Let's assume that LTS subjects are $n$ warehouses $S_1$, $S_2,\ldots,S_n$ between which cargo transportation is carried out. For cars loading and unloading in a warehouse $S_i$ $m_i$ brigades are created, $i=\overline{1,n}$. For simplicity we suggest that the same brigade is engaged in car unloading and in loading it afterwards with new production for further transportation so that the car's stay idle time was minimum. Transporting of the goods from one subject to another brings the latter some casual income and consequently the income of the first subject is reduced to this random variable (RV), however, or it can be vice versa, it doesn't matter for the model. Let's consider the dynamics of the network system $S_i$ incomes modification (warehouse $S_i$ of LTS). Let's designate by $V_i(t)$ system $S_i$ income at the moment $t$, and by $v_{i0}=V_i(0)$ its income at the initial moment. Then it is possible to present its income at the moment $t+\Delta t$ as the following:

$$V_i(t+\Delta t)=V_i(t)+\Delta V_i(t,\Delta t) \tag{11}$$

where $\Delta V_i(t,\Delta t)$ - the modification of income QS $S_i$ inn time interval $[t,t+\Delta t)$. To determinate this magnitude let's write out probable events which can happen in time $\Delta t$, and incomes modification of $S_i$ systems connected with these events.

1) With the probability $\lambda(t)p_{0i}\Delta t+o(\Delta t)$ the request to system $S_i$ will be received (to a warehouse $S_i$ car) which will bring some income, occupying space $r_{0i}$, where $r_{0i}$ - RV with expectation $M\{r_{0i}\}=a_{0i}$, $i=\overline{1,n}$.
2) With the probability $\mu_i(k_i(t))u(k_i(t))p_{i0}\Delta t+o(\Delta t)$ the request of $S_i$ will go to the outer world, thus income of QS $S_i$ will be reduced to the magnitude, occupying the of space $R_{i0}$, where $R_{i0}$ - RV from expectation $M\{R_{i0}\}=b_{i0}$, $i=\overline{1,n}.$
3) With the probability $\mu_j(k_j(t))u(k_j(t))p_{ji}\Delta t+o(\Delta t)$ the request $S_j$ system will pass to the $S_i$ system, thus $S_i$ system income will increase by magnitude, occupying the space of $r_{ji}$, and the $S_j$ income (i.e. the occupied area of the warehouse $S_j$) will decrease by this magnitude, where $r_{ji}$ - RV from expectation $M\{r_{ji}\}=a_{ji}$, $i,j=\overline{1,n}, i\neq j$.

4) With the probability $\mu_i(k_i(t))u(k_i(t))p_{ij}\Delta t + o(\Delta t)$ a request from system $S_i$ transits to system $S_j$, thus the income of $S_i$ will decrease by magnitude, occupying space $R_{ij}$ and the income of $S_j$ (i.e. the occupied square of the warehouse $S_j$) will increase by this magnitude, where $R_{ij}$ - RV from expectation $M\{R_{ij}\} = b_{ij}$, $i, j = \overline{1,n}$, $i \neq j$.

5) With the probability $1 - (\lambda(t)p_{0i} + \mu_i(k_i(t))u(k_i(t)) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \mu_j(k_j(t))u(k_j(t))p_{ji}) \times$ $\times \Delta t + o(t)$ at time interval $[t, t+\Delta t)$ the $S_i$ system state modification will not happen (i.e. the occupied square of the warehouse $S_i$ will not vary), $i = \overline{1,n}$.

Besides, for each small time interval $\Delta t$ system $S_i$ (subject $S_i$) increases the income by magnitude $r_i\,\Delta t$ by the means of percent on the money which is in its bank, where $r_i$ - RV from expectation $M\{r_i\} = c_i$, $i = \overline{1,n}$. We will consider also that RV $r_{ji}$, $R_{ij}$, $r_{0i}$, $R_{i0}$ are independent in relation to RV $r_i$, $i, j = \overline{1,n}$.

It is obvious that $r_{ji} = R_{ji}$ with probability 1, i.e.

$$a_{ji} = b_{ji},\ \ i, j = \overline{1,n} \tag{12}$$

Then from the mentioned above we can conclude the following:

$$\Delta V_i\ (t, \Delta t) = \begin{cases} r_{0i} + r_i\ \Delta t & \text{with probability} \quad \lambda(t)p_{0i}\Delta t + o(\Delta t), \\ -R_{i0} + r_i\ \Delta t & \text{with probability} \quad \mu_i(k_i(t))u(k_i(t))p_{i0}\Delta t + o(\Delta t), \\ r_{ji} + r_i\ \Delta t & \text{with probability} \quad \mu_j(k_j(t))u(k_j(t))p_{ji}\Delta t + o(\Delta t), \\ -R_{ij} + r_i\ \Delta t & \text{with probability} \quad \mu_i(k_i(t))u(k_i(t))p_{ij}\Delta t + o(\Delta t), \\ r_i\ \Delta t & \text{with probability} \quad 1 - \Big(\lambda(t)p_{0i} + \mu_i(k_i(t))u(k_i(t)) + \\ & \qquad + \sum_{\substack{j=1 \\ j\neq i}}^{n} \mu_j(k_j(t))u(k_j(t))p_{ji}\Big)\Delta t + o(\Delta t) \end{cases} \tag{13}$$

At the fixed realization of process $k(t)$, considering (13), it is possible to note:

$$M\{\Delta V_i(t,\Delta t)/k(t)\} = \left[\lambda(t)p_{0i}a_{0i} + c_i - \mu_i(k_i(t))u(k_i(t))\left(p_{i0}b_{i0} + \sum_{\substack{j=1\\ j\neq i}}^{n} p_{ij}b_{ij}\right) + \right.$$

$$\left. + \sum_{\substack{j=1\\ j\neq i}}^{n} \mu_j(k_j(t))u(k_j(t))p_{ji}a_{ji}\right]\Delta t + o(\Delta t)$$

Averaging on $k(t)$ taking into account a normalization state $\sum_k P(k(t)=k) = 1$ for modification of subject $S_i$ expected income we receive:

$$M\{\Delta V_i(t,\Delta t)\} = \sum_k P(k(t)=k)M\{\Delta V_i(t,\Delta t)/k(t)\} =$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty} P(k(t)=(k_1(t),k_2(t),\ldots,k_n(t)))\times$$

$$\times M\{\Delta V_i(t,\Delta t)/k(t)=(k_1(t),k_2(t),\ldots,k_n(t))\} =$$

$$= \left[\lambda(t)p_{0i}a_{0i} + c_i - \left(p_{i0}b_{i0} + \sum_{\substack{j=1\\ j\neq i}}^{n} p_{ij}b_{ij}\right)\sum_k P(k(t)=k)\mu_i(k_i(t))u(k_i(t)) + \right.$$

$$\left. + \sum_{\substack{j=1\\ j\neq i}}^{n} p_{ji}a_{ji}\sum_k P(k(t)=k)\mu_j(k_j(t))u(k_j(t))\right]\Delta t + o(\Delta t)$$

Let's consider that time intervals of servicing a request in the system $S_i$ (intervals of one car «unloadings - loadings» at the warehouse $S_i$) are distributed upon the demonstrative law with parameter $\mu_i$, $i=\overline{1,n}$, and $\lambda(t)=\lambda$. In this case

$$\mu_i(k_i(t)) = \begin{cases} \mu_i k_i(t), & k_i(t) \le m_i, \\ \mu_i m_i, & k_i(t) > m_i, \end{cases} \quad \mu_i(k_i(t))u(k_i(t)) = \mu_i \min(k_i(t), m_i), \ i=\overline{1,n}.$$

Let's assume also that the expression averaging $\mu_i(k_i(t))u(k_i(t))$ gives $\mu_i \min(N_i(t), m_i)$, i.e.

$$M \min(k_i(t), m_i) = \min(N_i(t), m_i) \tag{14}$$

where $N_i(t)$ is the average number of requests (expecting and served) in $S_i$ at the moment of time $t$, $i=\overline{1,n}$. Taking into account this supposition we receive the following approximate relation:

$$M\{\Delta V_i(t,\Delta t)\} = \left[ \lambda(t) p_{0i} a_{0i} + c_i - \mu_i \min(N_i(t), m_i) \left( p_{i0} b_{i0} + \sum_{\substack{j=1 \\ j\neq i}}^{n} p_{ij} b_{ij} \right) + \right.$$

$$\left. + \sum_{\substack{j=1 \\ j\neq i}}^{n} \mu_j \min(N_j(t), m_j) p_{ji} a_{ji} \right] \Delta t + o(\Delta t). \tag{15}$$

As an elementary stream of requests arrives in a network with intensity $\lambda$, i.e. the probability of $a$ requests inflow in $S_i$ for time $\Delta t$ looks like $P_a(\Delta t) = \frac{(\lambda p_{0i} \Delta t)^a}{a!} e^{-\lambda p_{0i} \Delta t}$, $a = 0,1,2,\ldots$, the average number of the requests which have arrived from the outside to $S_i$ for time $\Delta t$ is equal to $\lambda p_{0i} \Delta t$. We will find the average number of the occupied service lines in $S_i$ at the moment $t$, $i=\overline{1,n}$, by $\rho_i(t)$. Then $\mu_i \rho_i(t) \Delta t$ is the average number of the requests which have abandoned $S_i$ for time $\Delta t$, and $\sum_{\substack{j=1 \\ j\neq i}}^{n} \mu_j \rho_j(t) p_{ji} \Delta t$ is the average number of the requests which have arrived to $S_i$ from other subjects for time $\Delta t$. Therefore

$$N_i(t+\Delta t) - N_i(t) = \lambda p_{0i} \Delta t + \sum_{\substack{j=1 \\ j\neq i}}^{n} \mu_j \rho_j(t) p_{ji} \Delta t - \mu_i \rho_i(t) \Delta t, \ i=\overline{1,n}$$

Consequently, at the $\Delta t \to 0$ we receive the UDE system for $N_i(t)$:

$$\frac{dN_i(t)}{dt} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \mu_j \rho_j(t) p_{ji} - \mu_i \rho_i(t) + \lambda p_{0i}, \ i=\overline{1,n} \tag{16}$$

The precise magnitude $\rho_i(t)$ is impossible to discover and consequently it is approximated by the expression

$$\rho_i(t)=\begin{cases}N_i(t),\ N_i(t)\le m_i,\\ m_i, N_i(t)>m_i,\end{cases}=\min(N_i(t),\ m_i).$$

Then the set of equations (16) will become

$$\frac{dN_i(t)}{dt}=\sum_{\substack{j=1\\ j\ne i}}^{n}\mu_j p_{ji}\min(N_j(t),m_j)-\mu_i\min(N_i(t),m_i)+\lambda p_{0i},\ i=\overline{1,n} \tag{17}$$

It is a linear UDE system with the discontinious right members. It is necessary to solve it by space phase partition into a series of areas and solve each of them. The system (17) can be solved, for example, by the means of computer mathematics Maple 8 system.

Let's introduce a sign $v_i(t)=M\{V_i(t)\}$, $i=\overline{1,n}.$. From (11), (15) we receive

$$v_i\,(t+\Delta t)=v_i(t)+M\{\Delta V_i(t,\Delta t)\}=$$

$$=v_i(t)+\left[\lambda p_{0i}a_{0i}+c_i\ -\mu_i\min(N_i(t),\ m_i)\left(p_{i0}b_{i0}+\sum_{\substack{j=1\\ j\ne i}}^{n}p_{ij}b_{ij}\right)+\right.$$

$$\left.+\sum_{j=1}^{n}\mu_j\min(N_j(t),\ m_j)\,p_{ji}a_{ji}\right]\Delta t+o(\Delta t)\,.$$

Then, passing to a limit at $\Delta t\to 0$, we will receive inhomogeneous linear UDE of the first order

$$\frac{dv_i(t)}{dt}=-\mu_i\min(N_i(t),m_i)\left(p_{i0}b_{i0}+\sum_{j=1}^{n}p_{ij}b_{ij}\right)+$$

$$+\sum_{\substack{j=1\\ j\ne i}}^{n}\mu_j\min(N_j(t),m_j)\,p_{ji}a_{ji}+\lambda p_{0i}a_{0i}+c_i\ ,\ i=\overline{1,n} \tag{18}$$

Having set entry conditions $v_i(0)=v_{i0}$, $i=\overline{1,n}$, it is possible to discover expected incomes of network systems (average magnitudes of squares occupied in warehouses).

Knowing the expressions for $N_i(t)$ and the average warehouse squares occupied with cargoes, it is possible to design the warehouse squares of subjects $S_i$, $i = \overline{1,n}$. Let's consider the following modeling example.

**Example 2.** We will consider the closed network presented in figure 4, consisting of $n = 16$ one-linear QS, where $K = 75$ is the number of requests in the network. The requests service intensities in the network system lines are equal to: $\mu_1 = \mu_{13} = 2.7$, $\mu_6 = \mu_{15} = 3$, $\mu_4 = \mu_8 = \mu_{12} = \mu_{14} = 2.1$, $\mu_7 = \mu_{11} = \mu_{16} = 2$, $\mu_3 = \mu_9 = \mu_{10} = 4.5$, $\mu_2 = \mu_5 = 3.2$ and probabilities of request transitions between QS networks is $p_{15i} = 1/15$, $p_{i\,15} = 1$, $i = \overline{1,15}$, let's define also $p_{ii} = -1$, $i = \overline{1,16}$, remaining $p_{ij} = 0$, $i, j = \overline{1,16}$.

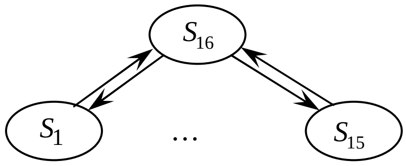


Fig. 4. The network scheme for example 2

Let's assume that the network functions so that on the average there are no queues observed in the peripheral QS, and the central QS functions in the conditions high workload. Then system DDE for the average number of requests in the network systems (16) will be represented as follows:

$$\frac{dN_i(t)}{dt} = \sum_{j=1}^{15} \mu_j p_{ji} N_j(t) + \mu_{16} p_{16i}, \; i = \overline{1,16} \tag{19}$$

Let's set values of income expectations from transitions between network states in the following way:

$$c_i = 7\sin\frac{\pi}{2(i+1)}, \; i = \overline{1,16}$$

$$a_{16\,i} = 0.5, \; i = \overline{1,6}, \; a_{16i} = 0.9, \; i = \overline{7,14}, \; a_{16\,15} = 1.55$$

$$a_{i\,16} = (9,11,18,20,28,10,8,15,10,9,12,17,4,13,5), \; i = \overline{1,15}$$

Then expected incomes of network systems discovered by the means of the package Mathematica 5.1 provided that at the initial instant of time $v_i(0) = 5, i = \overline{1,15}$, $v_{16}(0) = 50$, and entry conditions $N_i(0) = 4$, $i = 4,5,8$,

$N_i(0) = 2,\ i = 1,3,7,12$, $N_i(0) = 5,\ i = 2,13,14$, $N_i(0) = 3,\ i = 6,9,10,11,15$ $N_{16}(0) = 20$, behave as it is shown in Figure 5.

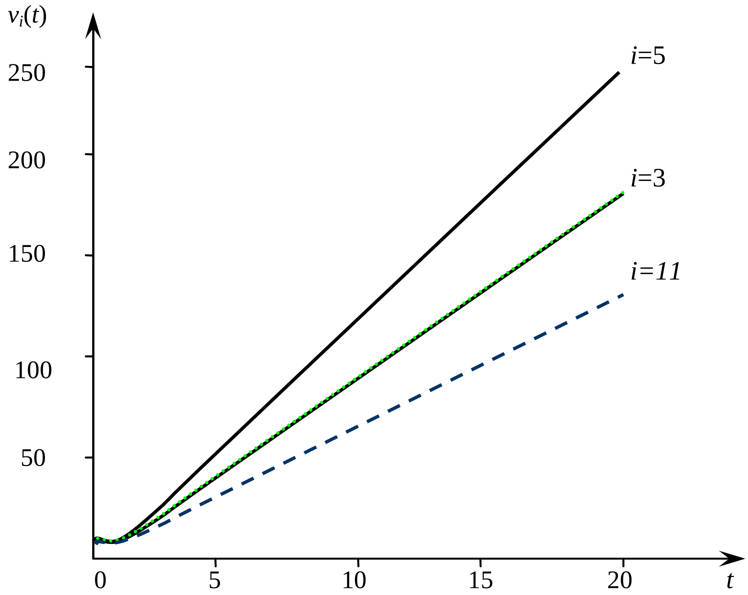


Fig. 5. Expected incomes of systems $S_3$, $S_5$, $S_{11}$

## Conclusions

Casual transport streams between various LTS subjects, casual duration of time intervals necessary for vehicle loading-unloading, casual volumes of transported cargoes and squares occupied with them in warehouse have predetermined the necessity of QN use for working out mathematical models of LTS functioning. In our report the new class of QN - Markov HM-networks with incomes is used for forecasting of expected incomes and squares of LTS subjects warehouse, and unlike previous researches such a case when the intensity of car stream arriving in LTS and the intensity of their service in subjects depend on time is considered.

The prospects of further work in the field is connected with the analysis methods development of arbitrary (non-Markov) webs with incomes and various peculiarities, control problems solution for them.

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