NUMERICAL MODELING OF HEAT TRANSFER IN A SINGLE BLOOD VESSEL AND SURROUNDING BIOLOGICAL TISSUE

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Abstract. The thermal interactions between the single blood vessel and surrounding biological tissue are analyzed. The temperature in the tissue is described by the Pennes equation, while the equation determining the change of blood temperature along the blood vessel is formulated on the basis of adequate energy balance. These equations are coupled by boundary condition given at the blood vessel wall. There are two models considered here in terms of blood vessel types. First is the supplying vessel model and the other one is traversing vessel model. Both are distinguished in the computations. The solution of the problem has been provided by means of finite difference method.

1. Governing equations

The biological tissue is heated by just one blood vessel [1-3] located at the central part of tissue cylinder, as shown in Figure 1.

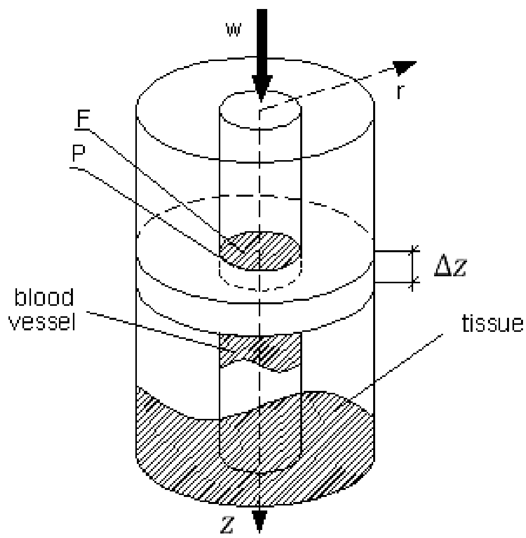

Fig. 1. Single blood vessel

Energy equation of this model for the volume $\Delta V = F\Delta z$, where F is the area of cross section of the vessel, can be written in the following manner [4, 5]:

$$Q_0 = H(z) - H(z + \Delta z) + Q_{Bmet}\Delta V \tag{1}$$

where Q_0 is a heat flowing from the vessel to the tissue, $H(z), H(z + \Delta z)$ is a blood enthalpy in $z, z + \Delta z$ respectively, Q_{Bmet} is a blood metabolic heat source.

Let us now define enthalpy

$$H(z) = wFc_B\rho_B T_B(z) \tag{2}$$

where w is a blood velocity, c_B [J/(m^{3}K)] - is a specific heat of blood referred to the volume (volumetric specific heat of blood), ρ_B [kg/m^3] is a blood density.

Using an Taylor expansion one has

$$H(z + \Delta z) = H(z) + \frac{\mathrm{d}H(z)}{\mathrm{d}z}\Delta z \tag{3}$$

or when we incorporate (2) into (3):

$$-H(z + \Delta z) + H(z) = -\frac{\mathrm{d}}{\mathrm{d}z}[wFc_B\rho_B T_B(z)]\Delta z \tag{4}$$

The heat Q_0 flowing between blood vessel and the tissue through area $P\Delta z$ (P is the circumference of the blood vessel) is equal to:

$$Q_0 = \alpha P\Delta z[T_B(z) - T_w(z)] \tag{5}$$

where α is a heat transfer coefficient between a blood vessel and a tissue, $T_w(z)$ is the wall temperature of the blood vessel.

Introducing the relations (4) and (5) into (1), we are getting the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}z}[wFc_B\rho_B T_B(z)]\Delta z + \alpha P[T_B(z) - T_w(z)] - Q_{Bmet}F = 0 \tag{6}$$

This is a standard differential equation which needs to be supplemented by the initial value: $T_B(0) = T_{B0}$.

Since the values w, F, c_B, ρ_B, P in the equation (6) are constant, we can rewrite this equation as follows

$$\frac{\mathrm{d}T_B(z)}{\mathrm{d}z} = \frac{\alpha P}{wFc_B\rho_B}[T_w(z) - T_B(z)] + \frac{Q_{Bmet}}{wc_B\rho_B} \tag{7}$$

Further, let us assume that the cross section of the blood vessel is a circle with a radius of R_1, then $F = \pi R_1^2, P = 2\pi R_1$ and the equation (7) can be rewritten into the following form:

$$\frac{\mathrm{d}T_\mathrm{B}(z)}{\mathrm{d}z} = \frac{2\alpha}{wc_\mathrm{B}\rho_\mathrm{B}R_1}[T_\mathrm{w}(R_1,z) - T_\mathrm{B}(z)] + \frac{Q_\mathrm{Bmet}}{wc_\mathrm{B}\rho_\mathrm{B}} \quad (8)$$

where $T_w(R_1,z) = T_w(z)$.

Let us now assume that the tissue temperature surrounding the vessel is changing only along the radius coordinate r and the heat transfer coefficient $\lambda[W/mK]$ is constant. Then the temperature distribution in the area describes the following equation:

$$R_1 < r < R_2: \quad \frac{\lambda}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T(r)}{\mathrm{d}r}\right) + c_\mathrm{B}\rho_\mathrm{B}G_\mathrm{B}[T_\mathrm{a} - T(r)] + Q_\mathrm{met} = 0 \quad (9)$$

where $G_B\,[1/s]$ is a perfusion coefficient, $Q_{met}\,[\mathrm{W/m^3}]$ is the metabolic heat source, T_a is a temperature in the artery.

On the wall of the blood vessel we assume boundary condition of 3rd type:

$$r = R_1: \quad \lambda\frac{\mathrm{d}T(r)}{\mathrm{d}r} = \alpha[T_w(R_1,z) - T_\mathrm{B}(z)] \quad (10)$$

whereas on the outer area of the tissue we apply Dirichlet condition:

$$r = R_2: \quad T(R_2) = T_t \quad (11)$$

where T_t is known temperature.

The equations (8) for the blood vessel and (9) for the tissue are coupled through unknown $T_w(R_1,z)$ and therefore the system of equations (8), (9) can be solved.

Let us introduce two vessel models

- supplying vessel model, in which we assume the following:

$$T_\mathrm{a} = T_\mathrm{B}(z) \quad (12)$$

  meaning that the blood temperature is substituted by temperature that exists in the cross section *z*.

- traversing vessel model, in which we assume the following:

$$T_\mathrm{a} = T_\mathrm{B0} \quad (13)$$

  meaning that the artery temperature is substituted by the temperature at the beginning of the blood vessel $z = 0$.

It is important to note that the length of the blood vessel z_{eq} (equivalent length) above which the temperature does not change is equal to the state of $\mathrm{d}T_\mathrm{B}/\mathrm{d}z = 0$.

2. Method of solution

The equation (8) that will be resolved here by means of the finite difference method, can be rewritten in the following manner:

$$0 < z < Z: \quad \frac{\mathrm{d}T_B(z)}{\mathrm{d}z} = A[T_w(R_1,z) - T_B(z)] + B \tag{14}$$

where

$$A = \frac{2\alpha}{wc_B\rho_B R_1}, \quad B = \frac{Q_{Bmet}}{wc_B\rho_B} \tag{15}$$

which is supplemented by the following initial condition:

$$z = 0: \quad T_B(z) = T_B(0) = T_{B0} \tag{16}$$

The equation (9) describing temperature in the tissue can be rewritten in the following form:

$$R_1 < r < R_2: \quad \lambda\frac{\mathrm{d}^2T(r)}{\mathrm{d}r^2} + \frac{\lambda}{r}\frac{\mathrm{d}T(r)}{\mathrm{d}r} + k[T_a - T(r)] + Q_{met} = 0 \tag{17}$$

where $k = c_B\rho_B G_B$. This equation is supplemented by the boundary conditions (10) and (11).

The equations (14) and (17), where the conjoined element is unknown temperature of wall blood vessel $T_w = T(R_1,z)$, are solved here using finite difference method. In this method the area $[R_1, R_2]$ is divided into n sections (Fig. 2).

$$R_1 = r_0 < r_1 < r_2 < r_3 < \cdots < r_i < r_{i+1} < \cdots < r_n = R_2 \tag{18}$$

where $r_i = R_1 + ih$, $i = 0,1,2,\dots,n$, where $h = (R_2 - R_1)/n$.

Next step is, analogically to r, to do a discretisation of variable z:

$$0 = z_0 < z_1 < z_2 < z_3 < \cdots < z_i < z_{i+1} < \cdots < z_n = Z \tag{19}$$

where $z_j = jh$, $j = 0,1,2,\dots,m$.

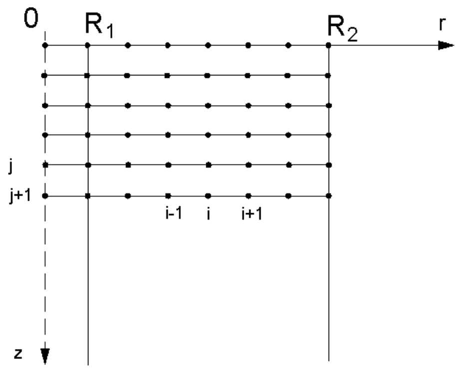

Fig. 2. Finite-difference mesh

The differential form of equation (17) for the inner node *i* is as follows

$$\lambda\frac{(T_{i-1}-2T_i+T_{i+1})}{h^2}+\frac{\lambda}{r_i}\frac{T_{i+1}-T_{i-1}}{2h}+k(T_a-T_i)+Q_{met}=0 \tag{20}$$

or

$$a_iT_{i-1}+bT_i+c_iT_{i+1}=d_i \tag{21}$$

where

$$a_i=\frac{\lambda}{h^2}-\frac{\lambda}{2hr_i},\quad b_i=-\left(\frac{2\lambda}{h^2}+k\right)$$
$$c_i=\frac{\lambda}{h^2}+\frac{\lambda}{2hr_i},\quad d_i=-kT_a-Q_{met} \tag{22}$$

In the equation (20) for a traversing blood vessel $T_a=T_{B0}$, and for supplying vessel $T_a=T_{Bj}$ which means it is a temperature for $z=z_j$, while for $j=0$ T_{B0} is known and described by the initial condition (16).

The differential form of boundary condition (10) can be written as follows

$$\lambda\frac{(T_1-T_0)}{R_1}=\alpha(T_0-T_{Bj}) \tag{23}$$

which is equal to:

$$-\left(\frac{\lambda}{R_1}+\alpha\right)T_0+\frac{\lambda}{R_1}T_1=-\lambda T_{Bj} \tag{24}$$

where the boundary condition (11) leads to the equation: $T_n=T_t$.

For the tissue subdomain we consider the following system of equations

$$\begin{bmatrix} -\left(\frac{\lambda}{R_1}+\alpha\right) & \frac{\lambda}{R_1} & & & & & \\ & a_1 & b & c_1 & & & \\ & & a_2 & b & c_2 & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & a_{n-1} & b & c_{n-1} \\ & & & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ \cdot \\ T_{n-1} \\ T_n \end{bmatrix} = \begin{bmatrix} -\alpha T_{Bj} \\ d_1 \\ d_2 \\ \cdot \\ d_{n-1} \\ d_n \end{bmatrix} \tag{25}$$

where blank spaces in the matrix are zeros.

Difference equation for equation (14) and *j* node has the following form:

$$\frac{T_{Bj+1}-T_{Bj}}{h} = A\left(T_{0j}-T_{Bj}\right)+B \tag{26}$$

and after transformations, we get the following equation:

$$T_{Bj+1} = (1-Ah)T_{Bj} + AhT_{0j} + Bh \tag{27}$$

where T_{0j} refers to the wall temperature in the blood vessel for $z = z_j$.

Concluding the steps necessary to solve the problem, for $j = 0$ we need to solve the system of equations (25) relating to the tissue. For $z = 0$ calculate $T_0, T_1, T_2, \dots, T_n$, next from equation (27) we need to calculate T_{B1} in the blood vessel and then to solve the equation (25) substituting T_{Bj} with T_{B1}. Doing so, we will calculate the temperatures in the nodes of the tissue for $z = h$. Then from the equation (27) we calculate the temperature T_{B2} and so forth.

Entry values

In these computations we assume the same thermo-physical blood and tissue properties, which are: $\lambda = \lambda_B = 0.5$ W/mK (thermal conductivities), $c_B = 3900$ J/(kgK) (blood volumetric specific heat), $\rho_B = 1060$ kg/m^3 (blood density). The radius of the blood vessel $R_1 = 0.002$ m, and the outer radius of the tissue surrounding the blood vessel $R_2 = 10R_1$.

Knowledge of the Nusselt number Nu = 4 allows us to calculate the heat transfer coefficient α between blood vessel and the surrounding tissue:

$$\text{Nu} = \frac{2R_1\alpha}{\lambda_B} \longrightarrow \alpha = \frac{\text{Nu}\lambda_B}{2R_1} \tag{28}$$

From the above equation we can calculate $\alpha = 500$ W/m^{2}K.

Blood velocity in the vessels can be calculated using Peclet number and the below equation:

$$\mathrm{Pe} = \frac{2R_1 c_B \rho_B w}{\lambda_B} \longrightarrow w = \frac{\mathrm{Pe}\lambda_B}{2R_1 c_B \rho_B} \tag{29}$$

For large vessels we can assume Pe = 100 and from the above equation we can calculate blood velocity traversing within the vessel: $w = 0.003\ \mathrm{m/s}$.

The calculations have been made under the assumption that Q_{met} = 1000 W/m^3 and Q_{Bmet} = 500 W/m^3 and the perfusion rate $w_B = G_B\rho_B$ = 10 kg/m^{2}s.

The entry blood temperature for $z = 0$ is assumed as $T_{B0} = 37°\mathrm{C}$. On the outer surface of the tissue surrounding the blood vessel $T_t = 37°\mathrm{C}$.

The problem has been solved using finite difference method, assuming the number $n = 20$ which means that $h = (R_2 - R_1)/n = 0.0009\ \mathrm{m}$.

4. Results of computations

Figures 3 and 4 show tissue temperature distribution in direction of radius *r* for coordinates $z = 0.1, 0.2, 0.3, 0.4, 0.5\ \mathrm{m}$, where Figure 3 refers to the traversing blood vessel $(T_a = T_{B0})$ and Figure 4 refers to the supplying blood vessel $(T_a = T_{Bj})$.

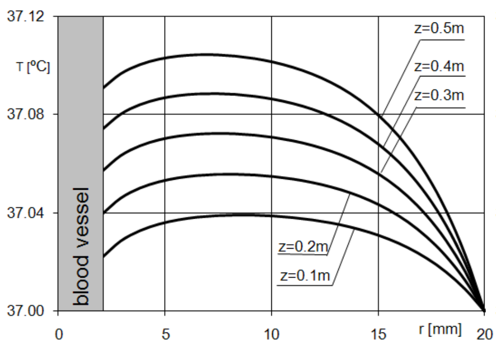

Fig. 3. Tissue temperature distribution for supplying vessel, where R_1 = 0.002

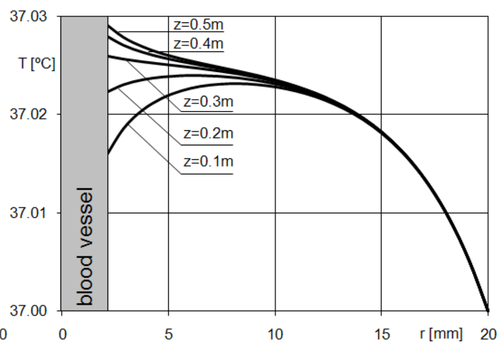

Fig. 4. Tissue temperature distribution for traversing vessel, where R_1 = 0.002

The distinctive difference in the both is that the tissue temperature for supplying vessel is higher than for traversing vessel. It is easy to observe from the Figure 4 that tissue temperature in supplying vessel for $r > 10$ is almost the same for all values of *z*. The computations have also included different radius of the vessel and the thickness of the tissue (Figures 6 and 7). Figure 7 and 8 describe a temperature difference in supplying and traversing blood vessel between different radius of the vessel and different thickness of the tissue. It is apparent from Figure 7 that the blood temperature for traversing vessel almost does not change for z > 0.4 m, which means it has reached equivalent balance status. On the other hand the equivalent balance status for supplying vessel would be much higher as we can see from the figure that the blood temperature for supplying vessel still rises.

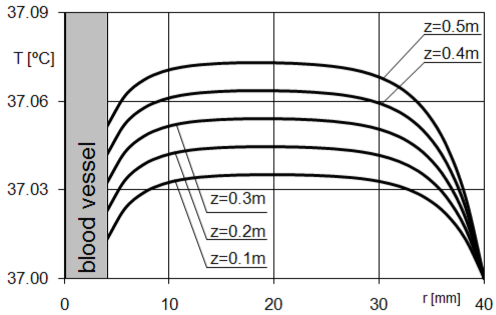

Fig. 5. Tissue temperature distribution for supplying vessel, where $R_1 = 0.004$

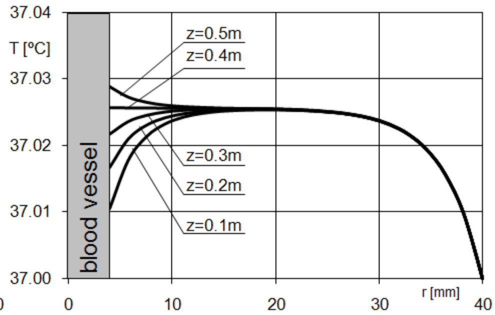

Fig. 6. Tissue temperature distribution for traversing vessel, where $R_1 = 0.004$

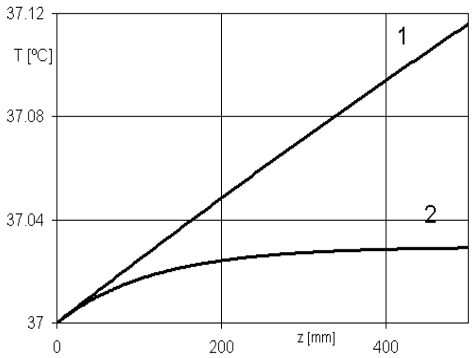

Fig. 7. Temperature distribution in the for $R_1 = 0.002$, $R_2 = 10$ R_1

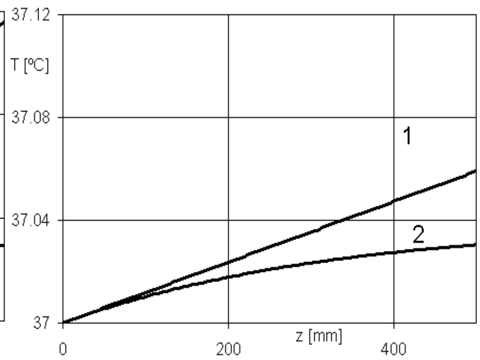

Fig. 8. Temperature distribution in vessel, the vessel for $R_1 = 0.004$, $R_2 = 10$ R_1

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