INTEGRATION ON HYPERSPHERES IN $R^n$

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Abstract. The subject of this paper are the hyperspheres in $n$-dimensional Euclidean space, which are the intersection of sphere and few planes. The paper concerns only the spheres with the center in the origin of coordinate system and the planes crossing through this point. Hypersphere parametrization and some integration formulas will be shown.

1. Explanation of notations used

In the beginning we would like to explain some of the symbols and theorems used in the first part of this paper.

Section of the vector $v = [v_1, v_2, ..., v_n] \in R^n$ from the first to $k$-th coefficients is the vector $\tilde{v}^k$ given by:

$$\tilde{v}^k = [v_1, v_2, ..., v_k] \in R^k$$  \hspace{1cm} (1)

Gramian matrix $G(a_1, a_2, ..., a_k)$ of elements $a_1, a_2, ..., a_k \in R^n$ is a symmetrical square matrix, which elements are the scalar products of corresponding vectors:

$$G(a_1, a_2, ..., a_k) = \begin{bmatrix}
\langle a_1, a_1 \rangle & \langle a_1, a_2 \rangle & \cdots & \langle a_1, a_k \rangle \\
\langle a_2, a_1 \rangle & \langle a_2, a_2 \rangle & \cdots & \langle a_2, a_k \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle a_k, a_1 \rangle & \langle a_k, a_2 \rangle & \cdots & \langle a_k, a_k \rangle
\end{bmatrix}$$  \hspace{1cm} (2)

Gramian determinant of elements $a_1, a_2, ..., a_k \in R^n$ is a determinant of gramian matrix $G(a_1, a_2, ..., a_k)$:
\[ G(a_1, a_2, \ldots, a_k) = \begin{bmatrix} \|a_1\|^2 & \langle a_1, a_2 \rangle & \cdots & \langle a_1, a_k \rangle \\ \langle a_1, a_2 \rangle & \|a_2\|^2 & \cdots & \langle a_2, a_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_1, a_k \rangle & \langle a_2, a_k \rangle & \cdots & \|a_k\|^2 \end{bmatrix} \] (3)

If vector \( b \) is orthogonal to every vector \( a_1, a_2, \ldots, a_k \) then

\[ G(b, a_1, a_2, \ldots, a_k) = \|b\|^2 G(a_1, a_2, \ldots, a_k) \] (4)

**Theorem 1.1 (of the norm of vector product)**

If \( c \) is a vector product of \( a_1, a_2, \ldots, a_k \in \mathbb{R}^n \) (\( c = a_1 \times a_2 \times \cdots \times a_{n-1} \)), then its norm is equal to square root of gramian determinant of \( a_1, a_2, \ldots, a_k \in \mathbb{R}^n \):

\[ \|c\| = \sqrt{G(a_1, a_2, \ldots, a_{n-1})} \] (5)

2. Parametrization of hyperspheres

Consider a set of \( k+1 \) equations with \( n \) variables:

\[
\begin{align*}
\sum_{i=1}^{k} x_i^2 + x_{k+1}^2 + \cdots + x_n^2 &= r^2 \\
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= 0 \\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= 0 \\
&\vdots \\
A_{k1}x_1 + A_{k2}x_2 + \cdots + A_{kn}x_n &= 0
\end{align*}
\] (6)

We assume that \( 2 \leq k \leq n-2 \) (when \( k = 1 \) refer to paper [3]) and

\[
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1k} \\
A_{21} & A_{22} & \cdots & A_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k1} & A_{k2} & \cdots & A_{kk}
\end{bmatrix} \neq 0
\] (7)

Let \( a_1 = [A_{11}, A_{12}, \ldots, A_{1n}] \), \( a_2 = [A_{21}, A_{22}, \ldots, A_{2n}] \), \ldots \( a_k = [A_{k1}, A_{k2}, \ldots, A_{kn}] \). Now we can begin solving the equation (6). We start by finding the set of vectors orthogonal to each other and orthogonal to every vector \( a_i \). This set will be a basis for the solutions of (6).
Step 1
In the first step, we calculate the \((k+1)\)-dimensional vector \(w_1\), according to formula:

\[
W_1 = \tilde{a}_1^{k+1} \times \tilde{a}_2^{k+1} \times ... \times \tilde{a}_k^{k+1}
\]  
(8)

Every coefficient of \(w_1\) can be calculated by the formula:

\[
w_{ij} = (-1)^{i-j}D_i
\]  
(9)

where \(D_i\) is a determinant created by omitting the \(i\)-th coefficient in every vector

\[
D = \begin{vmatrix}
A_{11} & A_{12} & \ldots & A_{1n} \\
A_{21} & A_{22} & \ldots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{i1} & A_{i2} & \ldots & A_{in}
\end{vmatrix}
\]  
(10)

According to theorem 1.1 the norm of vector \(w_1\) is equal to:

\[
\|w_1\| = \sqrt{G(\tilde{a}_1^{k+1}, \tilde{a}_2^{k+1}, \ldots, \tilde{a}_k^{k+1})}
\]  
(11)

Step 2
In the second step, we calculate the \((k+2)\)-dimensional vector \(w_2\), according to formula:

\[
w_2 = w_1 \times \tilde{a}_1^{k+2} \times \tilde{a}_2^{k+2} \times ... \times \tilde{a}_k^{k+2}
\]  
(12)

The norm of vector \(w_2\) is equal to:

\[
\|w_2\| = \sqrt{G(w_1, \tilde{a}_1^{k+2}, \tilde{a}_2^{k+2}, \ldots, \tilde{a}_k^{k+2})} = \|w_1\| \sqrt{G(\tilde{a}_1^{k+2}, \tilde{a}_2^{k+2}, \ldots, \tilde{a}_k^{k+2})}
\]  
(13)

Step \(n-k-1\)
In this step we calculate \((n-1)\)-dimensional vector

\[
w_{n-k-1} = w_{n-k-2} \times ... \times w_2 \times w_1 \times \tilde{a}_1^{\nu+1} \times \tilde{a}_2^{\nu+1} \times ... \times \tilde{a}_k^{\nu+1}
\]  
(14)

with the norm

\[
\|w_{n-k-1}\| = \|w_{n-k-2}\| \ldots \|w_2\| \|w_1\| \sqrt{G(\tilde{a}_1^{\nu+2}, \tilde{a}_2^{\nu+2}, \ldots, \tilde{a}_k^{\nu+2})}
\]  
(15)

Step \(n-k\)
Finally we calculate \(n\)-dimensional vector

\[
w_{n-k} = w_{n-k-1} \times ... \times w_2 \times w_1 \times a_1 \times a_2 \times ... \times a_k
\]  
(16)
with the norm

\[ \|w_{n-k}\| = \|w_{n-k-1}\| \ldots \|w_2\| \|w_1\| \sqrt{G(a_1, a_2, \ldots, a_k)} \] (17)

Next we replace the orthogonal set of vectors \( w_1, w_2, \ldots, w_{n-k} \) by the orthonormal set, according to the formula:

\[ v_i = \frac{\tilde{w}_i}{\|w_i\|} \] (18)

where \( \tilde{w}_i = [w_i, 0, \ldots, 0] \in \mathbb{R}^n \) (0 at \((n - k - i)\) -places).

Set of vectors \( v_i \) is a basis for solutions of (6). Let us define \( l = n - k \) as a codimension of our solution, and a matrix \( V \) as follows:

\[
V = \begin{bmatrix}
v_1 & v_2 & \cdots & v_n \\
v_2 & v_3 & \cdots & v_n \\
\vdots & \vdots & \ddots & \vdots \\
v_i & \cdots & & v_n \\
\end{bmatrix}
\] (19)

Now, let us recall an \( l \)-dimensional sphere parametrization:

\[
\begin{align*}
x_1 &= r \cos t_1 \cos t_2 \cos t_3 \ldots \cos t_{l-1} \\
x_2 &= r \sin t_1 \cos t_2 \cos t_3 \ldots \cos t_{l-1} \\
x_3 &= r \sin t_2 \cos t_3 \ldots \cos t_{l-1} \\
\vdots &= \vdots \\
x_l &= r \sin t_{l-1} \\
\end{align*}
\] (20)

where \( t_i \in [0, 2\pi] \), \( t_2, \ldots, t_{l-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \).

By implementing a vector function \( F(t) \):

\[
F(t) = \begin{bmatrix}
f_1(t) \\
f_2(t) \\
\vdots \\
f_{l-1}(t) \\
f_l(t) \\
\end{bmatrix} = \begin{bmatrix}
\cos t_1 \cos t_2 \cos t_3 \ldots \cos t_{l-1} \\
\sin t_1 \cos t_2 \cos t_3 \ldots \cos t_{l-1} \\
\vdots \\
\sin t_{l-1} \\
0 \\
0 \\
\end{bmatrix}
\] (21)
where \( t = [t_1, t_2, \ldots, t_{l-1}] \), we can simply write sphere parametrization as \( rF(t) \).

We calculate the parametrization of hypersphere by solving the following equation:

\[
V \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \end{bmatrix} = rF(t)
\]

(22)

Because matrix \( V \) is orthogonal, the solution of (6) is given by:

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \end{bmatrix} = V^T rF(t) = \begin{bmatrix} r(v_1 f_1(t) + v_{21} f_2(t) + \ldots + v_{1l} f_l(t)) \\ r(v_{12} f_1(t) + v_{22} f_2(t) + \ldots + v_{2l} f_l(t)) \\ \vdots \\ r(v_{nl} f_1(t) + v_{n2} f_2(t) + \ldots + v_{nl} f_l(t)) \\ \end{bmatrix}
\]

(23)

3. Notations and theorems used for integration on hyperspheres

Set of points given by (23) will be called an \( l \)-dimensional hypersphere and notated as \( S_l \):

\[
S_l = \begin{bmatrix} r(v_1 f_1(t) + v_{21} f_2(t) + \ldots + v_{1l} f_l(t)) \\ r(v_{12} f_1(t) + v_{22} f_2(t) + \ldots + v_{2l} f_l(t)) \\ \vdots \\ r(v_{nl} f_1(t) + v_{n2} f_2(t) + \ldots + v_{nl} f_l(t)) \\ \end{bmatrix}
\]

(24)

Partial derivative of a vector \( x \) given by (23) with respect to variable \( t_j \) is a following vector:

\[
\frac{\partial x}{\partial t_j} = \begin{bmatrix} \frac{\partial x_1}{\partial t_j} \\ \frac{\partial x_2}{\partial t_j} \\ \vdots \\ \frac{\partial x_n}{\partial t_j} \\ \end{bmatrix}
\]

(25)

where:

\[
\frac{\partial x}{\partial t_j} = r \left( v_1 \frac{df_1}{dt_j} + v_{21} \frac{df_2}{dt_j} + \ldots + v_{nl} \frac{df_l}{dt_j} \right)
\]

(26)
We should note as a fact, that

\[
\frac{\partial x}{\partial r} = \begin{cases} 
    r \cos t_{j+1} \cos t_{j+2} \ldots \cos t_{l-1} & \text{when } j < l - 1 \\
    r & \text{when } j = l - 1
\end{cases}
\]  

(27)

For every given matrix

\[
M = M_{k,n} = \begin{bmatrix} 
    m_{11} & m_{12} & \cdots & m_{1n} \\
    m_{21} & m_{22} & \cdots & m_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{k1} & m_{k2} & \cdots & m_{kn}
\end{bmatrix}
\]  

(28)

we can define a number \(|M|\), called this matrix module, by

\[
|M| = \sqrt{\sum_{k \leq j < h} \left| m_{j1} \ldots m_{jk} \right|^2}
\]  

(29)

**Theorem 3.1 (of matrix module, [1])**

Let \(M\) be a matrix given by (28), and let \(m_i = [m_{i1}, m_{i2}, \ldots, m_{in}]\) be the \(i\)-th row of that matrix. Then the matrix module is equal to square root of gramian matrix of its rows

\[
|M| = \sqrt{MM^T} = \sqrt{G(m_1, m_2, \ldots, m_n)}
\]  

(30)

**Theorem 3.2 (calculating the matrix determinant according to several rows, [1])**

Let \(k < n\) and \(l = n - k\). Then we have

\[
\begin{vmatrix} 
    a_{i1} & a_{i2} & \cdots & a_{in} \\
    \vdots & \ddots & \cdots & \vdots \\
    a_{k1} & a_{k2} & \cdots & a_{kn} \\
    b_{i1} & b_{i2} & \cdots & b_{in} \\
    \vdots & \ddots & \cdots & \vdots \\
    b_{l1} & b_{l2} & \cdots & b_{ln}
\end{vmatrix} = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \text{sgn}(i_1, \ldots, i_k, j_1, \ldots, j_l) \begin{vmatrix} 
    a_{i_1} & \cdots & a_{i_k} & b_{i_1} & \cdots & b_{i_k} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    a_{j_1} & \cdots & a_{j_l} & b_{j_1} & \cdots & b_{j_l}
\end{vmatrix}
\]  

(31)
Theorem 3.3 ([2], § 49, Tw. 9)

Let \( E \subset \mathbb{R}^n \), \( \gamma(E) \) be a hypersurface of parametrization \( \gamma : E \to \mathbb{R}^n \) and \( t \in \mathbb{R}^n \). If function \( f : \gamma(E) \to \mathbb{R} \) is integrable, then the following formula follows:

\[
\int_{\gamma(E)} f \, d\gamma(t) = \int_E (f \circ \gamma) \left| J(\gamma(t)) \right| \, dt
\]

(32)

4. Integration on hyperspheres

In the last part of this paper we develop the subject explained in [3], concerning integrals of the differential forms. We show the proof to the hypothesis shown there and expand it to the case, where the number of planes is greater than 1.

A function \( \omega \) given by

\[
\omega = \sum_{1 \leq i_1 < \cdots < i_{l-1} \leq n} f_{i_1,i_2,\ldots,i_{l-1}} \, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{l-1}}
\]

(33)

is called a differential form of \( l-1 \) order in \( \mathbb{R}^n \).

We define the integral from that form as follows:

\[
\int_{S_{n-1}} \omega = \sum_{1 \leq i_1 < \cdots < i_{l-1} \leq n} \int_{t \in \Omega} f_{i_1,i_2,\ldots,i_{l-1}}(x_1(t), x_2(t), \ldots, x_{l-1}(t)) \left| J_S(t_1,t_2,\ldots,t_{l-1}) \right| \, dt
\]

(34)

In the last paragraph of [3] there was a following hypothesis stated:

If \( \omega_{l-1} \) is a differential form given by

\[
\omega_{l-1} = \frac{(-1)^{n-1}}{(x_1^2 + \cdots + x_n^2)^{n-1}} \sum_{1 \leq i_1 < \cdots < i_{l-1} \leq n} sgn(i_1,i_2,\ldots,i_{l-1},j_1,j_2) \left| A_{i_1}^j \middle/ x_{i_1} \right| \wedge \left| A_{i_2}^j \middle/ x_{i_2} \right| \cdots \wedge \left| A_{i_{l-1}}^j \middle/ x_{i_{l-1}} \right|
\]

then the integral over hypersphere \( S_{n-1} \) is equal to

\[
\int_{S_{n-1}} \omega_{l-2} = \sigma_{l-1} \sqrt{A_1^2 + A_2^2 + \cdots + A_n^2}
\]

where \( \sigma_{l-1} \) denotes the measure of the surface of \( (l-1) \)-dimensional sphere of radius \( r = 1 \).

For every \( l \) the formula follows:

\[
\sigma_{l-1} = \frac{l \pi^{\frac{l}{2}}}{\Gamma\left(\frac{l}{2} + 1\right)}
\]

(35)
In paper [4], the hypothesis was extended and later proven, so the following theorem was introduced:

**Theorem (of integrating the differential forms over hypersphere)**

Let \( S \) be a \( l \)-dimensional hypersphere being an intersection of sphere in \( \mathbb{R}^n \) and \( k \) planes (that is \( S \) be a solution of (6)), and \( \omega_{-1} \) be a differential form given by

\[
\omega_{-1} = r^{-l} \sum_{l_0 \leq i_0 < \ldots < l_k \leq i_6} \text{sgn}(i_1, \ldots, i_{l+1}, j_1, \ldots, j_{k+1}) \begin{vmatrix} A_{i_1} & \ldots & A_{j_{k+1}} \\ \vdots & \ddots & \vdots \\ A_{i_1} & \ldots & A_{j_{k+1}} \\ x_{i_1} & \ldots & x_{j_{k+1}} \end{vmatrix} dx_{i_1} \wedge \cdots \wedge dx_{j_{k+1}}
\]

then the integral of \( \omega_{-1} \) over \( S \) is equal to

\[
\int_S \omega_{-1} = \sigma_{l-1} \sqrt{G(a_1, a_2, \ldots, a_k)}
\]

**Proof:**

Let \( t = (t_1, t_2, \ldots, t_{l-1}), \ \Omega = \{(t_1, t_2, \ldots, t_{l-1}) : 0 \leq t_1 \leq 2\pi, \frac{\pi}{2} \leq t_2 \leq \frac{\pi}{2}, \ldots, \frac{\pi}{2} \leq t_{l-1} \leq \frac{\pi}{2}\} \). According to theorem 3.3 we can bring integral over \( S \) to integral over \( \Omega \):

\[
\int_S \omega_{-1} = \int_{\Omega} r^{-l} \sum_{l_0 \leq i_0 < \ldots < l_k \leq i_6} \text{sgn}(i_1, \ldots, i_{l+1}, j_1, \ldots, j_{k+1}) \Theta dt
\]

where

\[
\Theta = \begin{vmatrix} A_{i_1} & \ldots & A_{j_{k+1}} \\ \vdots & \ddots & \vdots \\ A_{i_1} & \ldots & A_{j_{k+1}} \\ x_{i_1} & \ldots & x_{j_{k+1}} \end{vmatrix}
\]

According to theorem 3.2, equation (38) can be simply written as:
From the theorem 3.1 it follows, that the integral above can be written as

\[
\int_{S_{n-1}} \omega_{n-1} = r^{-1} \int_{\Omega} \left| \begin{array}{ccc}
\frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\
\vdots & \vdots & & \vdots \\
\frac{\partial x_1}{\partial t_{n-1}} & \frac{\partial x_2}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \\
x_1 & x_2 & \cdots & x_n
\end{array} \right| dt
\]

(40)

Every vector \( x, \frac{\partial x}{\partial t_1}, \frac{\partial x}{\partial t_2}, \ldots, \frac{\partial x}{\partial t_{n-1}} \) is a linear combination of \( v_1, v_2, \ldots, v_l \), so following the property of gramian determinant (see (4)), we can write:

\[
\int_{S_{n-1}} \omega_{n-1} = r^{-1} \int_{\Omega} \left| \begin{array}{ccc}
\frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} & \cdots & \frac{\partial x}{\partial t_{n-1}} \\
\vdots & \vdots & & \vdots \\
\frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} & \cdots & \frac{\partial x}{\partial t_{n-1}} \\
1 & \omega \cdots A_n
\end{array} \right| dt
\]

(41)

By inserting the values of vector norms, we obtain (see (27)):

\[
\left\| \frac{\partial x}{\partial t_1} \right\| \left\| \frac{\partial x}{\partial t_2} \right\| \cdots \left\| \frac{\partial x}{\partial t_{n-1}} \right\| = r^l \cos t_2 \cos^2 t_3 \cdots \cos^{l-2} t_{n-1}
\]

(42)

Inserting this value to our integral yields:

\[
\int_{S_{n-1}} \omega_{n-1} = \sqrt{G(a_1, a_2, \ldots, a_k)} \int_{\Omega} \cos t_2 \cos^2 t_3 \cdots \cos^{l-2} t_{n-1} dt
\]

(44)

Calculating the following integral brings the proof to an end:

\[
\int_{\Omega} \cos t_2 \cos^2 t_3 \cdots \cos^{l-2} t_{n-1} dt = \frac{l! \pi^l}{\Gamma(l+1)} = \sigma_{l-1}
\]

(45)

\[
\int_{S_{n-1}} \omega_{n-1} = \sigma_{l-1} \sqrt{G(a_1, a_2, \ldots, a_k)}
\]

(46)
References

[3] Ligus M., Całkowanie na hiperokrągach w $R^n$, praca dyplomowa w Instytucie Matematyki Politechniki Częstochowskiej.