ON HOMOGENEITY OF THE TANGENCY RELATION OF ARCS

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Abstract. In this paper the problem of the homogeneity of some tangency relation of sets for the rectifiable arcs in the generalized metric spaces is considered. Some sufficient conditions for the homogeneity of this relation are given.

Introduction

Let (E, l) be a generalized metric space (see [1]). In system (E, l) symbol l denotes a non-negative real function defined on Cartesian product $E_0 \times E_0$ of family E_0 of all non-empty subsets of set E.

Let l_0 be the function of the form:

$$l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for} \quad x, y \in E \tag{1}$$

By some assumptions relating to function l, function l_0 defined by formula (1) will be the metric of set E.
We will denote by $S_l(p, r)_u$ (see [2, 3]) the u-neighbourhood of sphere $S_l(p, r)$ in space (E, l) defined by the following formula:

$$S_l(p, r)_u = \begin{cases} \bigcup\limits_{q \in S_l(p,r)} K_l(q, u) & \text{for} \quad u > 0 \\ S_l(p, r) & \text{for} \quad u = 0 \end{cases} \tag{2}$$

Let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0 \tag{3}$$

We say that pair (A, B) of sets $A, B \in E_0$ is (a, b)-clustered at point p of space (E, l), if 0 is the cluster point of the set of all real numbers $r > 0$ such that the sets $A \cap S_l(p, r)_{a(r)}$, $B \cap S_l(p, r)_{b(r)}$ are non-empty.

Let $T_l(a,b,k,p)$ (see [1, 3, 4]) be the tangency relation of sets in generalized metric space (E,l) defined by formula:

$$T_l(a,b,k,p) = \{(A,B) : \ A,B \in E_0, \text{ pair } (A,B) \text{ is } (a,b)\text{-clustered at point } p \text{ of space } (E,l) \text{ and } \frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0\} \tag{4}$$

If pair $(A,B) \in T_l(a,b,k,p)$, then we say that set $A \in E_0$ is (a,b)-tangent of order $k > 0$ to set $B \in E_0$ at point p of the generalized metric space (E,l).

Let ρ be a metric of set E and let A, B be arbitrary sets from family E_0. Let us define:

$$\rho(A,B) = \inf\{\rho(x,y) : x \in A, \ y \in B\} \tag{5}$$

$$d_\rho A = \sup\{\rho(x,y) : x,y \in A\} \tag{6}$$

We shall denote by $\mathfrak{F}_\rho$ the class of all functions l fulfilling the conditions:

1^0 $l : E_0 \times E_0 \longrightarrow [0,\infty)$,

2^0 $\rho(A,B) \le l(A,B) \le d_\rho(A \cup B)$ for $A,B \in E_0$.

From equality (1) and from condition 2^0 it follows that

$$l(\{x\},\{y\}) = l_0(x,y) = \rho(x,y) \ \text{ for } \ l \in \mathfrak{F}_\rho \ \text{ and } \ x,y \in E \tag{7}$$

The above equality implies that any function $l \in \mathfrak{F}_\rho$ generates metric ρ in set E.

Let $\widetilde{A}_p$ be the class of the rectifiable arcs with the origin at point $p \in E$ of the form (see [5, 6]):

$$\widetilde{A}_p = \{A \in E_0 : \lim_{A \ni x \to p} \frac{\ell(\breve{px})}{\rho(p,x)} = g < \infty\} \tag{8}$$

where $\ell(\breve{px})$ denotes the length of the arc $\breve{px}$ with ends p and x.

In paper [7] we considered the problem of the additivity of the tangency relation $T_l(a,b,k,p)$ in class of arcs $\widetilde{A}_p$ in generalized metric space (E,l), where $l \in \mathfrak{F}_\rho$ (see also considerations related to the problem of the tangency of arcs in paper [8]).

If we assume in Corollary 1.2 of Theorem 1.1 of paper [7] that functions $l_1, l_2, ..., l_m \in \mathfrak{F}_\rho$ are equal to function $l \in \mathfrak{F}_\rho$, then

$$(A,B)\in T_{ml}(a,b,k,p) \iff (A,B)\in T_l(a,b,k,p) \tag{9}$$

for $A,B\in\widetilde{A}_p$, $m\in\mathbf{N}$, and for functions a,b fulfilling the condition

$$\frac{a(r)}{r}\xrightarrow[r\to 0^+]{}0 \quad\text{and}\quad \frac{b(r)}{r}\xrightarrow[r\to 0^+]{}0 \tag{10}$$

Therefore arises the question: is the equivalence (9) true for an arbitrary $m\in\mathbf{R}_+$? The answer to this question is positive, what will be proved in the present paper.

Definition 1. *We shall call the tangency relation $T_l(a,b,k,p)$ homogeneous of order* 0 *in class $\mathfrak{F}_\rho$, if equivalence* (9) *holds for arbitrary $m>0$, $l\in\mathfrak{F}_\rho$ and $A,B\in E_0$.*

In this paper the problem of the homogeneity of tangency relation $T_l(a,b,1,p)$ in the class of functions $\mathfrak{F}_\rho$ for arcs of the class $\widetilde{A}_p$ is considered. Some sufficient conditions for the homogeneity of order 0 of this tangency relation of arcs $\widetilde{A}_p$ will be given in Section 1.

1. Homogeneity of the tangency relation of arcs

We shall define:

$$(ml)(A,B)=ml(A,B)\ \text{ for }\ m>0,\ l\in\mathfrak{F}_\rho\ \text{ and }\ A,B\in\widetilde{A}_p \tag{11}$$

Let $S_l(p,r)_u$ be the u-neighbourhood of sphere $S_l(p,r)$ in space (E,l) defined by formula (2). For this set we shall prove the following lemma:

Lemma 1.1. *If $l\in\mathfrak{F}_\rho$, then*

$$S_{ml}(p,r)_u=S_l(p,r/m)_{u/m}\quad \textit{for}\quad m>0 \tag{12}$$

`Proof.` Using (11) we have

$$\begin{aligned} S_{ml}(p,r)&=\{x\in E:\ (ml)(\{p\},\{x\})=r\}\\ &=\{x\in E:\ ml(\{p\},\{x\})=r\}=\{x\in E:\ l(\{p\},\{x\})=r/m\}\\ &=S_l(p,r/m),\end{aligned}$$

i.e.

$$S_{ml}(p,r)=S_l(p,r/m)\ \text{ for }\ l\in\mathfrak{F}_\rho\ \text{ and }\ m>0 \tag{13}$$

Analogously

$$K_{ml}(p,r)=K_l(p,r/m)\ \text{ for }\ l\in\mathfrak{F}_\rho\ \text{ and }\ m>0 \tag{14}$$

From (13), (14) and from definition (4) of set $S_l(p,r)_u$ we get thesis of this lemma.

The following lemma was proved in paper [3]

Lemma 1.2. *If non-decreasing function a fulfils the condition*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \tag{15}$$

then for arbitrary set $A \in A^*_{p,k}$ having the Darboux property at point p of space (E,ρ) and $m > 0$

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p, r/m)_{a(r)/m}) \xrightarrow[r\to 0^+]{} 0 \tag{16}$$

As class of arcs $\widetilde{A}_p$ is a subset of class $A^*_{p,1}$ of sets having the Darboux property at point p of space (E,ρ) (see [3]), then from this lemma immediately follows:

Corollary 1.1. *If non-decreasing function a fulfils the condition*

$$\frac{a(r)}{r} \xrightarrow[r\to 0^+]{} 0 \tag{17}$$

then for an arbitrary arc $A \in \widetilde{A}_p$ and $m > 0$

$$\frac{1}{r} d_\rho(A \cap S_\rho(p, r/m)_{a(r)/m}) \xrightarrow[r\to 0^+]{} 0 \tag{18}$$

Theorem 1.1. *If non-decreasing functions a, b fulfil the condition*

$$\frac{a(r)}{r} \xrightarrow[r\to 0^+]{} 0 \quad \textit{and} \quad \frac{b(r)}{r} \xrightarrow[r\to 0^+]{} 0 \tag{19}$$

then tangency relation $T_l(a,b,1,p)$ is homogeneous of order 0 in class of arcs $\widetilde{A}_p$ for any function $l \in \mathfrak{F}_\rho$.

`Proof.` Let us assume that $(A,B) \in T_{ml}(a,b,1,p)$ for $A, B \in \widetilde{A}_p$. Hence it follows

$$\frac{1}{r}(ml)(A \cap S_{ml}(p,r)_{a(r)}, B \cap S_{ml}(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0.$$

Hence, from (11) and from Lemma 1.1 we obtain

$$\frac{1}{r} l(A \cap S_l(p, r/m)_{a(r)/m}, B \cap S_l(p, r/m)_{b(r)/m}) \xrightarrow[r\to 0^+]{} 0 \tag{20}$$

From (20) and from the fact that $l \in \mathfrak{F}_\rho$ it results

$$\frac{1}{r}\rho(A\cap S_l(p,r/m)_{a(r)/m}, B\cap S_l(p,r/m)_{b(r)/m}) \xrightarrow[r\to 0^+]{} 0.$$

Hence and from Theorem 2 of paper [4] on the compatibility of the tangency relations of rectifiable arcs we get

$$\frac{1}{r}\rho(A\cap S_l(p,r/m)_{a(r)}, B\cap S_l(p,r/m)_{b(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{21}$$

If $0 < m < 1$, then from the definition of set $S_l(p,r)_u$ and from the assumption that a and b are non-decreasing functions below the inequality

$$\begin{aligned} 0 &\le \rho(A\cap S_l(p,r/m)_{a(r/m)}, B\cap S_l(p,r/m)_{b(r/m)}) \\ &\le \rho(A\cap S_l(p,r/m)_{a(r)}, B\cap S_l(p,r/m)_{b(r)}). \end{aligned}$$

Hence, from (21) and from the properties of function $l\in\mathfrak{F}_\rho$ we obtain

$$\frac{1}{r}\rho(A\cap S_l(p,r/m)_{a(r/m)}, B\cap S_l(p,r/m)_{b(r/m)}) \xrightarrow[r\to 0^+]{} 0.$$

Hence and from Theorem 1 of paper [4] on the compatibility of the tangency relations of rectifiable arcs we have

$$\frac{1}{r}d_\rho((A\cap S_l(p,r/m)_{a(r/m)})\cup(B\cap S_l(p,r/m)_{b(r/m)})) \xrightarrow[r\to 0^+]{} 0.$$

Hence and from the fact that $l\in\mathfrak{F}_\rho$

$$\frac{1}{r}l(A\cap S_l(p,r/m)_{a(r/m)}, B\cap S_l(p,r/m)_{b(r/m)}) \xrightarrow[r\to 0^+]{} 0,$$

whence it follows

$$\frac{1}{t}l(A\cap S_l(p,t)_{a(t)}, B\cap S_l(p,t)_{b(t)}) \xrightarrow[t\to 0^+]{} 0 \tag{22}$$

From (21) and from Theorem 1 of paper [4] it results

$$\frac{1}{r}d_\rho((A\cap S_l(p,r/m)_{a(r)})\cup(B\cap S_l(p,r/m)_{b(r)})) \xrightarrow[r\to 0^+]{} 0 \tag{23}$$

If $m\ge 1$, then from (23) and from the assumption on functions a, b we get

$$\frac{1}{r}d_\rho((A\cap S_l(p,r/m)_{a(r/m)})\cup(B\cap S_l(p,r/m)_{b(r/m)})) \xrightarrow[r\to 0^+]{} 0 \tag{24}$$

Hence and from the fact that $l\in\mathfrak{F}_\rho$ it follows

$$\frac{1}{r}l(A\cap S_l(p,r/m)_{a(r/m)}, B\cap S_l(p,r/m)_{b(r/m)}) \xrightarrow[r\to 0^+]{} 0,$$

which yields condition (22).

Because pair (A,B) of arcs $A,B\in\widetilde{A}_p$ is (a,b)-clustered at point p of space (E,l), then from (22) it follows that $(A,B)\in T_l(a,b,1,p)$ for $A,B\in\widetilde{A}_p$.

Now we assume that $(A,B)\in T_l(a,b,1,p)$ for $A,B\in\widetilde{A}_p$. Hence it follows that

$$\frac{1}{r}l(A\cap S_l(p,r/m)_{a(r/m)},B\cap S_l(p,r/m)_{b(r/m)})\xrightarrow[r\to 0^+]{}0.$$

Hence and from the fact that $l\in\mathfrak{F}_\rho$ we obtain

$$\frac{1}{r}\rho(A\cap S_l(p,r/m)_{a(r/m)},B\cap S_l(p,r/m)_{b(r/m)})\xrightarrow[r\to 0^+]{}0 \tag{25}$$

Hence and from Theorem 1 of paper [4] we have

$$\frac{1}{r}d_\rho((A\cap S_l(p,r/m)_{a(r/m)})\cup(B\cap S_l(p,r/m)_{b(r/m)}))\xrightarrow[r\to 0^+]{}0 \tag{26}$$

If $0<m<1$, then from the fact that a,b are non-decreasing functions it follows

$$\begin{aligned}0&\le d_\rho((A\cap S_l(p,r/m)_{a(r)})\cup(B\cap S_l(p,r/m)_{b(r)}))\\&\le d_\rho((A\cap S_l(p,r/m)_{a(r/m)})\cup(B\cap S_l(p,r/m)_{b(r/m)})).\end{aligned}$$

From here and from (26) we get

$$\frac{1}{r}d_\rho((A\cap S_l(p,r/m)_{a(r)})\cup(B\cap S_l(p,r/m)_{b(r)}))\xrightarrow[r\to 0^+]{}0 \tag{27}$$

Hence and from Theorem 2 of paper [4] we have

$$\frac{1}{r}d_\rho((A\cap S_l(p,r/m)_{a(r)/m})\cup(B\cap S_l(p,r/m)_{b(r)/m}))\xrightarrow[r\to 0^+]{}0,$$

from where

$$\frac{1}{r}l(A\cap S_l(p,r/m)_{a(r)/m},B\cap S_l(p,r/m)_{b(r)/m})\xrightarrow[r\to 0^+]{}0,$$

i.e.

$$\frac{1}{r}(ml)(A\cap S_{ml}(p,r)_{a(r)},B\cap S_{ml}(p,r)_{b(r)})\xrightarrow[r\to 0^+]{}0 \tag{28}$$

If $m\ge 1$, then from the fact that a,b are the non-decreasing functions we get the inequality

$$\begin{aligned}0&\le \rho(A\cap S_l(p,r/m)_{a(r)},B\cap S_l(p,r/m)_{b(r)})\\&\le \rho(A\cap S_l(p,r/m)_{a(r/m)},B\cap S_l(p,r/m)_{b(r/m)}).\end{aligned}$$

Hence and from (25) we have

$$\frac{1}{r}\rho(A\cap S_l(p,r/m)_{a(r)}, B\cap S_l(p,r/m)_{b(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{29}$$

From (29), from Theorem 1 and Theorem 2 (see also Corollary 2) of paper [4] we obtain

$$\frac{1}{r}d_\rho((A\cap S_l(p,r/m)_{a(r)/m})\cup(B\cap S_l(p,r/m)_{b(r)/m})) \xrightarrow[r\to 0^+]{} 0.$$

Hence and from the fact that $l\in\mathfrak{F}_\rho$ we get

$$\frac{1}{r}l(A\cap S_l(p,r/m)_{a(r)/m}),(B\cap S_l(p,r/m)_{b(r)/m}) \xrightarrow[r\to 0^+]{} 0,$$

what gives the condition (28).

Because pair (A,B) of arcs $A,B\in\widetilde{A}_p$ is (a,b)-clustered at point p of space (E,ml), then from the condition (28) it follows that $(A,B)\in T_{ml}(a,b,1,p)$ for $A,B\in\widetilde{A}_p$. This ends the proof of the theorem.

Let $A,B\in E_0$ and $l_1,l_2,\dots,l_n$ be arbitrary functions belonging to class $\mathfrak{F}_\rho$. Let us define the sum of tangency relations (see [2]):

$$(A,B)\in\bigcup_{i=1}^{n}T_{l_i}(a,b,k,p)\Leftrightarrow(A,B)\in T_{l_j}(a,b,k,p)\quad\text{for any}\quad j\in\{1,2,\dots,n\}.$$

Hence, from Theorem 1.1 and from Theorem 2.1 on the additivity of the tangency relation $T_l(a,b,k,p)$ of paper [2] we get

Corollary 1.2. *If non-decreasing functions a,b fulfil condition* (19) *and $l,l_1,l_2,...,l_n\in\mathfrak{F}_\rho$, then*

$$(A,B)\in T_{m_1l_1+\cdots+m_nl_n}(a,b,1,p) \iff (A,B)\in T_{l_j}(a,b,1,p) \tag{30}$$

for an $j\in\{1,2,...,n\}$, for arbitrary $A,B\in\widetilde{A}_p$, and $m_1,...,m_n>0$.

Let A_p be the class of the rectifiable arcs with the Archimedean property at point p of metric space (E,ρ).

We say that rectifiable arc A has the Archimedean property at point p of space (E,ρ) iff

$$\lim_{A\ni x\to p}\frac{\ell(\breve{px})}{\rho(p,x)}=1 \tag{31}$$

where $\ell(\breve{px})$ denotes the length of arc $\breve{px}$.

Because the class A_p is contained in the class of arcs $\widetilde{A}_p$, then from Theorem 1.1 of this paper follows:

Corollary 1.3. *If non-decreasing functions a, b fulfil condition* (19)*, then tangency relation $T_l(a,b,1,p)$ is homogeneous of order* 0 *in the class of functions $\mathfrak{F}_f$ for arcs of the class A_p fulfilling condition* (31).

References

[1] Waliszewski W., On the tangency of sets in generalized metric spaces, Ann. Polon. Math. 1973, 28, 275-284.

[2] Konik T., On some tangency relation of sets, Publ. Math. Debrecen 1999, 55/3-4, 411-419.

[3] Konik T., On some property of the tangency relation of sets, Balkan Journal of Geometry and Its Applications 2007, 12(1), 76-84.

[4] Konik T., On the compatibility of the tangency relations of rectifiable arcs, Scientific Research of the Institute of Mathematics and Computer Science of Czestochowa University of Technology 2007, 1(6), 103-108.

[5] Waliszewski W., On the tangency of sets in a metric space, Colloq. Math. 1966, 15, 127-131.

[6] Gołąb S., Moszner Z., Sur le contact des courbes dans les espaces metriques généraux, Colloq. Math. 1963, 10, 105-311.

[7] Konik T., Additivity of the tangency relation of rectifiable arcs, Scientific Research of the Institute of Mathematics and Computer Science of Czestochowa University of Technology 2009, 2(8), 87-94.

[8] Pascali E., Tangency and ortogonality in metric spaces, Demonstratio Math. 2005, 38(2), 437-449.