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ON DIFFERENTIABLE SOLUTIONS FOR ONE-TERM NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Two one-term nonlinear fractional differential equations with the left- or right-sided Caputo derivative are discussed. The existence and uniqueness of solutions, generated by the respective stationary function, is proved in the space of continuously differentiable function. The proof, based on the Banach theorem, includes the extension of the Bielecki method of equivalent norms.

Introduction

Non-integer order operators are now applied in mathematical modelling in many areas of mechanics, physics, control theory, engineering, bioengineering, economics and chemistry (see monographs [1-8] and the references therein). The theory of such operators, fractional calculus, describes derivatives and integrals of non-integer order as well as their properties. In applications of fractional calculus, a new class of integral-differential equations called fractional differential equations (FDE), has been developed. The methods of solving FDE extend differential equations theory and include fixed point theorems, integral transform methods as well as operational methods based on properties of new classes of special functions [6-16]. In the paper we shall consider two one-term nonlinear fractional differential equations. The differential part contains the Caputo left- or right-sided derivative. We reformulate the equations in terms of a mapping determined on a space of continuously differentiable functions. In proof of the existence of a solution, we apply the fixed point theorem and an extended version of the Bielecki method of equivalent norms [17]. The obtained result is global in the sense that the construction is valid for an arbitrary finite interval.

The paper is divided into two main parts. In Section 1 we gather all the necessary definitions and properties of the operators from fractional calculus. There we also introduce a family of norms indexed by a non-negative real parameter and a non-negative vector function. Then we prove their equivalence in the space of continuously differentiable functions. The existence-uniqueness results are included in Section 2, where we obtain solutions for one-term nonlinear FDE containing the left- or the right-sided Caputo derivative. The paper is closed by a short discussion of possible extension of the presented method of proof.

1. Preliminaries

We recall here some of the definitions of non-integer order operators and their properties. We start with integrals defined for functions determined on finite interval $[a,b]$ (compare monographs [8],[20]).

Definition 1.1
Riemann-Liouville integrals of order *α*, denoted as $I^{\alpha}_{a+} f(t), I^{\alpha}_{b-} f(t)$, are given by the following formulas for $\mathrm{Re}(\alpha) > 0$:

$$I^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(u)du}{(t-u)^{1-\alpha}} \qquad t > a \tag{1}$$

$$I^{\alpha}_{b-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(u)du}{(u-t)^{1-\alpha}} \qquad t < b\,. \tag{2}$$

The first of the above integrals is called the left-sided Riemann-Liouville integral and the next, the right-sided integral respectively. Applying defined fractional integrals, we can construct fractional derivatives. In our paper we shall consider one-term nonlinear FDE with Caputo derivatives given in the definition below.

Definition 1.2
Caputo derivatives of order *α*, denoted as ${}^{c}D^{\alpha}_{a+}$ and ${}^{c}D^{\alpha}_{b-}$ for $\mathrm{Re}(\alpha) \in (n-1, n)$, look as follows:

$${}^{c}D^{\alpha}_{a+} f(t) = \left(\frac{d}{dt}\right)^{n} I^{n-\alpha}_{a+} f(t) \qquad t > a \tag{3}$$

$${}^{c}D^{\alpha}_{b-} f(t) = \left(-\frac{d}{dt}\right)^{n} I^{n-\alpha}_{b-} f(t) \qquad t < b \tag{4}$$

Similar to the integrals defined in (1), (2) we have the left-sided derivative (3) and the right-sided derivative (4).

A detailed review of the properties and applications of non-integer order operators can be found in monographs [4-9, 20]. We quote here two composition rules for integrals and Caputo derivatives. Further, we shall apply them in the transformation of fractional differential equations and in the investigation of their solutions.

Property 1.3
The following composition rules hold for any $t \in [a,b]$:

$${}^{c}D^{\alpha}_{a+} I^{\alpha}_{a+} f(t) = f(t) \tag{5}$$

$$ {}^{c}D^{\alpha}_{b-} I^{\alpha}_{b-} f(t) = f(t) \,, \tag{6} $$

provided function f is continuous i.e. $f \in C[a,b]$

Property 1.4
The following composition rules hold for any $t \in [a,b]$ and $\operatorname{Re}(\beta) > \operatorname{Re}(\alpha)$:

$$ {}^{c}D^{\alpha}_{a+} I^{\beta}_{a+} f(t) = I^{\beta-\alpha}_{a+} f(t) \tag{7} $$

$$ {}^{c}D^{\alpha}_{b-} I^{\beta}_{b-} f(t) = I^{\beta-\alpha}_{b-} f(t) \,, \tag{8} $$

provided function f is continuous i.e. $f \in C[a,b]$.

Definition 1.5
Function space $C^m[a,b]$ is a space of *m*-times continuously differentiable functions determined by the condition

$$ C^m[a,b] = \{x \in C[a,b]; \quad x^{(m)} \in C[a,b]\} \,. $$

The above space, endowed with a metric induced by the following norm:

$$ \| f \|^{C^m} = \sum_{j=0}^{m} \sup_{t \in [a,b]} \left| f^{(j)}(t) \right| \tag{9} $$

is a metric and complete space.
Norm (9), standard for the $C^m[a,b]$ space, can be modified so as to be useful in the proof of existence-uniqueness of solutions for the discussed FDE.

Definition 1.6
We introduce the following new norm on function space $C^m[a,b]$

$$ \| f \|^{C^m}_{\kappa} = \sum_{j=0}^{m} \sup_{t \in [a,b]} \left| f^{(j)}(t) \right| e^{-\kappa G_j(t)} \tag{10} $$

where G_j are arbitrary continuous, non-negative functions and κ is a positive real number.

Let us note that for $\kappa = 0$ we recover norm (9) and the corresponding induced metric.

It is easy to check that for any value of parameters $\kappa_1, \kappa_2 \in R_+ \cup \{0\}$ norms (10) are equivalent to each other on the $C^m[a,b]$ space. This fact also implies their equivalence to the standard norm $\| \cdot \|^{C^m}$.

Property 1.7

Norms $\|\cdot\|_{\kappa_1}^{C^m}$ and $\|\cdot\|_{\kappa_2}^{C^m}$ are equivalent on space $C^m[a,b]$ for any $\kappa_1,\kappa_2 \in R_+ \cup \{0\}$ and functions G_j obeying the conditions of Definition 1.6.

Proof: let us assume $\kappa_1 < \kappa_2$. Then the following inequalities are valid for exponential coefficients

$$e^{-\kappa_1 G_j(t)} \geq e^{-\kappa_2 G_j(t)} \qquad j = 0,1,\ldots,m$$

$$\sum_{j=0}^{m} \sup_{t\in[a,b]} \left| f^{(j)}(t) \right| e^{-\kappa_1 G_j(t)} \geq \sum_{j=0}^{m} \sup_{t\in[a,b]} \left| f^{(j)}(t) \right| e^{-\kappa_2 G_j(t)}$$

As a consequence we obtain the relations for any function $f \in C^m[a,b]$ and $\kappa_1 < \kappa_2$

$$\|f\|_{\kappa_1}^{C^m} \geq \|f\|_{\kappa_2}^{C^m} .$$

Due to the properties of the exponential function we also have

$$\|f\|_{\kappa_1}^{C^m} = \sum_{j=0}^{m} \sup_{t\in[a,b]} \left| f^{(j)}(t) \right| e^{-\kappa_1 G_j(t)} = \sum_{j=0}^{m} \sup_{t\in[a,b]} e^{(\kappa_2-\kappa_1)G_j(t)} \sup_{t\in[a,b]} \left| f^{(j)}(t) \right| e^{-\kappa_2 G_j(t)} \leq$$

$$\leq \sum_{j=0}^{m} B_j \sup_{t\in[a,b]} \left| f^{(j)}(t) \right| e^{-\kappa_2 G_j(t)} \leq B \sum_{j=0}^{m} \sup_{t\in[a,b]} \left| f^{(j)}(t) \right| e^{-\kappa_2 G_j(t)} = B\|f\|_{\kappa_2}^{C^m}$$

where we denoted constants B_j and B as

$$B_j = \sup_{t\in[a,b]} e^{(\kappa_2-\kappa_1)G_j(t)} \qquad B = \max\{B_1,\ldots,B_m\}.$$

From our calculations, it follows that for any function $f \in C^m[a,b]$, its norms fulfill the inequalities:

$$B\|f\|_{\kappa_2}^{C^m} \geq \|f\|_{\kappa_1}^{C^m} \geq \|f\|_{\kappa_2}^{C^m}$$

which means norms $\|\cdot\|_{\kappa_1}^{C^m}$ and $\|\cdot\|_{\kappa_2}^{C^m}$ are equivalent on function space $C^m[a,b]$.

2. Main results

In this section we shall solve one-term nonlinear FDE in the form of

$$ {}^{c}D^{\alpha}_{a+} f(t) = \Psi(t, f(t)) \tag{11} $$

$$ {}^{c}D^{\alpha}_{b-} f(t) = \Psi(t, f(t)) \tag{12} $$

The first of the above equations contains the left-sided Caputo derivative, the second one the right-sided Caputo derivative. We assume that in both cases order α is a real number and equations are determined on arbitrary finite interval $[a,b]$.

Let us observe that equations (11), (12) can be reformulated as the following equivalent fractional integral equations

$$ f(t) = I^{\alpha}_{a+}\Psi(t, f(t)) \ + \varphi_o(t) \tag{13} $$

$$ f(t) = I^{\alpha}_{b-}\Psi(t, f(t)) \ + \overline{\varphi}_o(t) \tag{14} $$

where functions φ_o and $\overline{\varphi}_o$ are arbitrary stationary functions of the left- and right--sided Caputo derivative. In both cases these functions are polynomials of degree determined by the order of the respective derivative.

Further, the obtained fractional integral equation (13) is an example of a more general integral equation:

$$ f(t) = \int_a^t K_0(t,s)\Psi(s, f(s))ds \ + \varphi_o(t) \tag{15} $$

with the kernel given in our case as

$$ K_0(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} & s \le t \\ 0 & s > t. \end{cases} \tag{16} $$

We use kernel K_0 to construct a mapping on the $C^{n-2}[a,b]$ space:

$$ Ty(t) := \int_a^t K_0(t,s)\Psi(s, y(s))ds \ + \varphi_o(t)\,. \tag{17} $$

Now, we are able to rewrite equations (11), (13, (15) as the following fixed point condition

$$ f(t) = Tf(t) $$

determined on the space of *n–2*-times continuously differentiable functions.
In what follows we shall also apply kernels in the form of

$$K_j(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-j-1}}{\Gamma(\alpha-j)} & s \le t \\ 0 & s > t. \end{cases} \tag{18}$$

Let us note that kernels (16), (18) are non-negative, continuous functions on set $[a,b]\times[a,b]$ when $\alpha \ge 1$ and $j = 0,1,...,n-2$.

The procedure of transforming an FDE of type (11) into the above fixed point condition with mapping (17) was discussed earlier in the fractional calculus. Our aim is to present an efficient method of proof that it is contractive in the chosen function space. In the lemma below we extend the Bielecki method of equivalent norms [17] and apply the family of norms indexed by a non-negative scalar parameter and non-negative vector function, defined in (10).

Lemma 2.1

Mapping *T*, defined in formula (16) is contractive on the $(C^{n-2}[a,b], \ \|\cdot\|_{\kappa}^{C^{n-2}})$ space when $\kappa > \sum_{j=0}^{n-2} M_j$, and

$$G_j(t) = \int_a^t K_j(u)L(u)du \tag{19}$$

$$K_j(u) = \sup_{t\in[a,b]} K_j(t,u).$$

Proof: let us denote

$$\|y\|_{\kappa}^{0} := \sup_{t\in[a,b]} |y(t)| \cdot e^{-\kappa G_0(t)}$$

and in addition constants

$$M_0 = 1 \qquad M_j = \sup_{t\in[a,b]} e^{-\kappa G_j(t)+\kappa G_0(t)} \qquad j = 1,...,n-2. \tag{20}$$

The distance between images *Tx* and *Ty* obeys the following inequalities for any pair of functions $x, y \in C^{n-2}[a,b]$

$$\|Tx - Ty\|_{\kappa}^{C^{n-2}} = \sum_{j=0}^{n-2} \sup_{t\in[a,b]} e^{-\kappa G_j(t)} \left| \int_a^t K_j(t,s)\Psi(s,x(s))ds - \int_a^t K_j(t,s)\Psi(s,y(s))ds \right| \le$$

$$\leq \sum_{j=0}^{n-2} \sup_{t\in[a,b]} e^{-\kappa G_j(t)} \int_a^t \left| K_j(t,s)\left[\Psi(s,x(s)) - \Psi(s,y(s))\right] \right| ds \leq$$

$$\leq \sum_{j=0}^{n-2} \sup_{t\in[a,b]} e^{-\kappa G_j(t)} \int_a^t K_j(s)L(s)\left|x(s)-y(s)\right| ds =$$

$$= \sum_{j=0}^{n-2} \sup_{t\in[a,b]} e^{-\kappa G_j(t)} \int_a^t K_j(s)L(s)\left|x(s)-y(s)\right| e^{-\kappa G_j(s)} e^{\kappa G_j(s)} ds \leq$$

$$= \sum_{j=0}^{n-2} \sup_{t\in[a,b]} e^{-\kappa G_j(t)} \int_a^t K_j(s)L(s)\left|x(s)-y(s)\right| e^{-\kappa G_0(s)} e^{\kappa G_0(s)} e^{-\kappa G_j(s)} e^{\kappa G_j(s)} ds \leq$$

$$\leq \frac{1}{\kappa}\left\|x-y\right\|_\kappa^0 \cdot \sum_{j=0}^{n-2} M_j \sup_{t\in[a,b]} e^{-\kappa G_j(t)} \int_a^t \kappa K_j(s)L(s) e^{\kappa G_j(s)} ds =$$

$$= \frac{1}{\kappa}\left\|x-y\right\|_\kappa^0 \cdot \sum_{j=0}^{n-2} M_j \sup_{t\in[a,b]} e^{-\kappa G_j(t)}\left(e^{\kappa G_j(t)} - 1\right) \leq \frac{1}{\kappa}\left\|x-y\right\|_\kappa^0 \cdot \sum_{j=0}^{n-2} M_j \sup_{t\in[a,b]}\left(1 - e^{-\kappa G_j(t)}\right) \leq$$

$$\leq \frac{\sum_{j=0}^{n-2} M_j}{\kappa}\left\|x-y\right\|_\kappa^0 \leq \frac{\sum_{j=0}^{n-2} M_j}{\kappa}\left\|x-y\right\|_\kappa^{C^{n-2}}$$

Summarizing, we observe that for any pair of functions $x, y \in C^{n-2}[a,b]$, the distance between their images is smaller than the initial distance, provided we consider mapping T acting on the $(C^{n-2}[a,b], \|\cdot\|_\kappa^{C^{n-2}})$ space with $\kappa > \sum_{j=0}^{n-2} M_j$:

$$\left\|Tx - Ty\right\|_\kappa^{C^{n-2}} \leq \frac{\sum_{j=0}^{n-2} M_j}{\kappa}\left\|x-y\right\|_\kappa^{C^{n-2}}$$

where M_j are given in (20) for $j = 0,...,n-2$.

The proved property of mapping T leads to the following proposition describing the solution of a one-term FDE with the left-sided Caputo derivative.

Proposition 2.2

If $\alpha \geq 1$, $\alpha \in (n-1,n)$ and function Ψ fulfills the following Lipschitz condition

$$\left|\Psi(t,x(t)) - \Psi(t,y(t))\right| \leq L(t)\left|x(t)-y(t)\right| \qquad t\in[a,b] \qquad \forall x,y \in C[a,b]$$

then each stationary function φ_0 of the left-sided Caputo derivative, generates a unique $C^{n-2}[a,b]$ solution of fractional differential equation

$$ {}^{c}D^{\alpha}_{a+} f(t) = \Psi(t, f(t)) $$

This solution is a limit of iterations of mapping T defined below on the $C^{n-2}[a,b]$ space:

$$ Ty(t) := I^{\alpha}_{a+}\Psi(t, y(t)) + \varphi_o(t) \qquad y \in C^{n-2}[a,b] $$

$$ f = \lim_{k\to\infty} (T)^k \psi $$

where $\psi \in C^{n-2}[a,b]$ arbitrary.

Proof: we start the proof by transforming equation (11) into an equivalent fractional integral equation. Thanks to the composition rule given in Property 1.3 we can rewrite equation (11) as follows

$$ f(t) = I^{\alpha}_{a+}\Psi(t, f(t)) + \varphi_o(t) $$

where stationary function φ_0 of the left-sided Caputo derivative fulfills the condition:

$$ {}^{c}D^{\alpha}_{a+}\varphi_0(t) = 0 $$

As is known from fractional calculus, the class of the stationary functions considered as a subclass of continuous functions contains only polynomials of degree dependent on the order of derivative. When we assume $\alpha \in (n-1, n)$, then each φ_0 is a polynomial of degree *n-1* with arbitrary coefficients.

Hence, we can reformulate FDE (11) and the corresponding fractional integral equation (13) as the following fixed point condition:

$$ f(t) = Tf(t) \qquad t \in [a,b] \tag{21} $$

where mapping T is generated by φ_0 and defined as follows for any given $y \in C^{n-2}[a,b]$

$$ Ty(t) := I^{\alpha}_{a+}\Psi(t, y(t)) + \varphi_o(t) $$

In the next step we apply Property 1.4 and note

$$ (Ty(t))^{(j)} = I^{\alpha-j}_{a+}\Psi(t, y(t)) + \varphi_0^{(j)}(t) \qquad j = 1, \dots, n-2 $$

where fractional integrals on the right-hand side are determined by the kernels given in (18)

$$(Ty(t))^{(j)} = \int_a^t K_j(t,s)\Psi(s, y(s))ds + \varphi_0^{(j)}(t) \qquad j = 1,\dots,n-2$$

As fractional integrals are bounded in the space of functions continuous in interval $[a,b]$ (compare Lemma 2.8 in [8]), we see that mapping T transforms functions from the $C^{n-2}[a,b]$ space into images belonging to the same function space

$$T : C^{n-2}[a,b] \to C^{n-2}[a,b]$$

Applying Lemma 2.1 we observe that the above mapping is contractive on the $(C^{n-2}[a,b], \|\cdot\|_{\kappa}^{C^{n-2}})$ space, provided κ is large enough. Therefore function $f \in C^{n-2}[a,b]$, obeying fixed point condition (21), exists. This function is a unique solution to equation (11) in this space, generated by φ_0. In addition, the Banach theorem allows us to construct this unique solution as a limit of iterations of mapping *T.*

Let us note that an analogous existence-uniqueness result can be proved for equation (12) with the right-sided Caputo derivative. As we know [20], the left- and right-sided derivatives on finite interval $[a,b]$ are connected by the action of reflection operator *Q:*

$$^{c}D_{b-}^{\alpha} f(t) = Q\,^{c}D_{a+}^{\alpha} Qf(t)$$

$$Qf(t) := f(a+b-t).$$

Thus, we can rewrite equation (12) as an FDE with the left-sided Caputo derivative. Its solution exists due to Proposition 2.2 and is generated by the respective polynomial $Q\overline{\varphi}_0$.

Proposition 2.3
If $\alpha \geq 1$, $\alpha \in (n-1,n)$ and function Ψ fulfills the following Lipschitz condition

$$\left|\Psi(t,x(t)) - \Psi(t,y(t))\right| \leq L(t)\left|x(t) - y(t)\right| \qquad t \in [a,b] \qquad \forall x,y \in C[a,b],$$

then each stationary function $\overline{\varphi}_0$ of the right-sided Caputo derivative, generates a unique $C^{n-2}[a,b]$ solution of fractional differential equation

$$^{c}D_{b-}^{\alpha} f(t) = \Psi(t, f(t)).$$

This solution is a limit of iterations of mapping T defined below on the $C^{n-2}[a,b]$ space:

$$\overline{T}y(t) := I_{b-}^{\alpha}\Psi(t,y(t)) + \overline{\varphi}_o(t) \qquad y \in C^{n-2}[a,b]$$

$$f = \lim_{k\to\infty}(\overline{T})^k \psi$$

where $\psi \in C^{n-2}[a,b]$ arbitrary.

3. Final remarks

Two types of nonlinear one-term FDE with Caputo derivatives were discussed and solved globally in an arbitrary finite interval. We derived their solution in the class of continuously differentiable functions. As a main tool of proof we applied the extended version of the Bielecki method and explicitly constructed solutions generated by stationary functions of the Caputo derivative.

Let us note that the scaling of norms using exponential functions, as in formula (10), restricts our results to equations of fractional order $\alpha \geq 1$. The obtained solutions belong to the $C^{n-2}[a,b]$ space when $\alpha \in (n-1,n)$. In further investigations we shall also consider case $0<\alpha<1$ and in general for $\alpha \in (n-1,n)$ solutions in the corresponding $C^{n-1}[a,b]$ space. To this aim we propose to apply scaling via the Mittag-Leffler function which is a generalization of the exponential function:

$$\| f \|_{\kappa}^{C^{n-1}} = \sum_{j=0}^{n-1} \sup_{t\in[a,b]} \frac{\left| f^{(j)}(t) \right|}{E_{\alpha-j,1}(\kappa(t-a)^{\alpha-j})}$$

$$E_{\alpha-j,1}(\kappa(t-a)^{\alpha-j}) = \sum_{k=0}^{\infty} \frac{\kappa^k (t-a)^{(\alpha-j)k}}{\Gamma((\alpha-j)k+1)}.$$

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