POSITIVE AND BOUNDED BELOW SOLUTIONS FOR CERTAIN NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Two types of one-term nonlinear fractional differential equations are considered and the existence of solutions in the space of continuous, positive and bounded below functions is proved. We transform an equation containing the left- or right-sided Caputo derivative into a fixed point condition and apply the Banach theorem and extended Bielecki method of equivalent norms.

Introduction

Fractional calculus involves derivatives and integrals of non-integer order in addition to the classical ones of integer-order. Contrary to the traditional name, the order of operators can be a fraction, an arbitrary real or complex number. Such operators are now an integral part of mathematical modelling methods in many areas of mechanics, physics, control theory, bioengineering, economics and chemistry (see monographs [1-8] and the references therein). In the applications of fractional calculus, a new class of integral-differential equations has been developed. They include integrals and derivatives of non-integer order and in general the higher order equations contain compositions of fractional derivatives. Solutions have been studied for two decades [6-16] and the methods of solving include fixed point theorems, integral transform methods as well as operational methods based on properties of new classes of special functions. We shall study here one-term fractional differential equations (FDE) which means the differential part includes only one fractional derivative which in the considered case is a Caputo left- or right-sided one. In the paper we apply the Bielecki method of equivalent norms [17] (compare also [18, 19]) as a main tool of proving the existence-uniqueness of the solutions and extend it to the FDE with a right-sided operator.

The paper is organized as follows. In the next section we recall the definitions and some properties of fractional operators. We also introduce the family of function spaces of continuous and bounded below functions, determined in an arbitrary finite interval. On these spaces two types of norms are constructed following Bielecki's ideas [17]. They depend on a scaling positive parameter and on a nonnegative continuous function. Being equivalent to the standard supremum norm, they yield the same convergent sequences and the same limits. In Section 2 we consider two types of nonlinear integral equations: containing the left-sided integral (as is standard in the Volterra equations theory) or the right-sided one. Using the norms introduced in Section 1 and the induced metrics we prove the existence and uniqueness of the solutions for both types of equations. The main results of the paper are included in Section 3, where we transform certain nonlinear FDE into equivalent integral ones. Next, we apply the results of Section 2 to construct the solutions generated by the respective stationary functions of Caputo derivatives.

1. Preliminaries

We recall here some of the operators of fractional calculus. We start with integrals defined for functions determined on finite interval [8, 20].

Definition 1.1

Riemann-Liouville integrals of order α, denoted as $I^{\alpha}_{0+} f(t), I^{\alpha}_{b-} f(t)$, are given by the formulas below for $\mathrm{Re}(\alpha) > 0$:

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u)du}{(t-u)^{1-\alpha}} \qquad t > 0 \tag{1}$$

$$I^{\alpha}_{b-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(u)du}{(u-t)^{1-\alpha}} \qquad t < b \tag{2}$$

The first of the above integrals is called the left-sided Riemann-Liouville integral and the next, the right-sided integral respectively. Having defined fractional integrals, we can construct fractional derivatives. In our paper we shall consider one-term fractional differential equations with Caputo derivatives given in the following definition.

Definition 1.2

Caputo derivatives of order α, denoted as ${}^{c}D^{\alpha}_{0+}$ and ${}^{c}D^{\alpha}_{b-}$ for $\mathrm{Re}(\alpha) \in (n-1, n)$, look as follows:

$${}^{c}D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(u)}{(t-u)^{\alpha-n+1}} \qquad t > 0 \tag{3}$$

$${}^{c}D^{\alpha}_{b-} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(u)}{(u-t)^{\alpha-n+1}} \qquad t < b \tag{4}$$

Similar to the integrals defined in (1), (2) we have the left-sided derivative (3) and the right-sided derivative (4).

An extensive review of the properties and applications of the presented operators can be found in monographs [6-8, 20]. We only quote the composition rules for integrals and derivatives. We shall apply them further in the transformation of the fractional differential equations discussed in our paper.

Property 1.3
The following composition rules hold for any $t \in [0,b]$:

$$ {}^{c}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t) \tag{5} $$

$$ {}^{c}D_{b-}^{\alpha} I_{b-}^{\alpha} f(t) = f(t)\,, \tag{6} $$

provided function f is continuous i.e. $f \in C[0,b]$.
Our aim is to study nonlinear FDE on a finite interval in the form of

$$ {}^{c}D_{0+}^{\alpha} x(t) = \lambda \cdot x(t)^{r} \tag{7} $$

$$ {}^{c}D_{b-}^{\alpha} x(t) = \lambda \cdot x(t)^{r} \tag{8} $$

and to find their positive solutions belonging to the function space given in the definition below.

Definition 1.4
Function space $C_{m_0,g}[0,b]$ is a subspace of the space of continuous functions determined by the condition

$$ C_{m_0,g}[0,b] = \{x \in C[0,b]; \quad x(t) \geq m_0 g(t) > 0 \quad \forall\, t \in [0,b]\}\,. $$

Let us note that the above space, endowed with a metric induced by the supremum norm, is a metric and complete space.
The nonlinear terms on the right-hand side of equations (7), (8) fulfill the following Lipschitz-type condition.

Lemma 1.5
Let $x, y \in C_{m_0,g}[0,b]$ be a pair of arbitrary functions. Then the following inequality is valid for any $t \in [0,b]$

$$ \left| x(t)^{r} - y(t)^{r} \right| \leq L(t) \left| x(t) - y(t) \right| \tag{9} $$

where function coefficient L looks as follows

$$ L(t) = \frac{1}{l(m_0 g(t))^{1-r}} \tag{10} $$

for $r \in (\frac{1}{n}, \frac{1}{l})$ or $-r \in (\frac{1}{n}, \frac{1}{l})$ in the case of negative *r*, where $l, n \in N$.

Proof: we observe that for any $r \in (0,1)$ a pair of integer numbers exists such that

$$n^{-1} < r < l^{-1} \qquad l, n \in N \tag{11}$$

Let us assume that $x(t) \geq y(t)$ for given $t \in [0,b]$. We shall prove inequality (9) in this case only as all the calculations are analogous when $x(t) \leq y(t)$. Starting from the left-hand side of (9) we obtain

$$\left|x(t)^r - y(t)^r\right| = \left|x(t)^r\right| \cdot \left|1 - \left(\frac{y(t)}{x(t)}\right)^r\right| \leq \left|x(t)^r\right| \cdot \left|1 - \left(\frac{y(t)}{x(t)}\right)^{1/l}\right| =$$

$$= \frac{\left|x(t)^r\right| \cdot \left|1 - \frac{y(t)}{x(t)}\right|}{\left|\sum_{k=0}^{l-1} \left(\frac{y(t)}{x(t)}\right)^{k/l}\right|} \leq \frac{\left|x(t)^{r-1}\right| \cdot \left|x(t) - y(t)\right|}{\sum_{k=0}^{l-1} \left(\frac{y(t)}{x(t)}\right)^{k/l}} = \frac{\left|x(t) - y(t)\right|}{\sum_{k=0}^{l-1} y(t)^{k/l}\, x(t)^{(l-rl-k)/l}},$$

where we applied the formula for partial sums of the geometric series. As functions x and y belong to the $C_{m_0,g}[0,b]$ space, we arrive at inequality (9):

$$\left|x(t)^r - y(t)^r\right| \leq \frac{\left|x(t) - y(t)\right|}{\sum_{k=0}^{l-1} \left|y(t)^{k/l}\right| \left|x(t)^{(l-rl-k)/l}\right|} \leq \frac{\left|x(t) - y(t)\right|}{l\left(m_0 g(t)\right)^{1-r}}.$$

Now, we consider the case when exponent r is negative. Similar to the first part of our calculations we assume $x(t) \geq y(t)$, $r \in (-1,0)$ and r fulfills the condition

$$n^{-1} < -r < l^{-1} \qquad l, n \in N.$$

Then we get

$$\left|x(t)^r - y(t)^r\right| = \left|\frac{1}{x(t)^{-r}} - \frac{1}{y(t)^{-r}}\right| = \left|x(t)^r\, y(t)^r\right| \cdot \left|y(t)^{-r} - x(t)^{-r}\right| \leq$$

$$\leq \left|y(t)^r\right| \cdot \left|1 - \left(\frac{y(t)}{x(t)}\right)^{1/l}\right| = \left|y(t)^r\right| \cdot \frac{\left|1 - \left(\frac{y(t)}{x(t)}\right)^{1/l}\right|}{\left|1 - \frac{y(t)}{x(t)}\right|} \cdot \left|1 - \frac{y(t)}{x(t)}\right| =$$

$$= \frac{\left|y(t)^r\right| \cdot \left|1 - \frac{y(t)}{x(t)}\right|}{\sum_{k=0}^{l-1} \left|\left(\frac{y(t)}{x(t)}\right)^{k/l}\right|} = \frac{|x(t) - y(t)|}{\sum_{k=0}^{l-1} x(t)^{(l-k)/l} y(t)^{(k-rl)/l}}.$$

Remembering that $x(t) \geq m_0 g(t)$, $y(t) \geq m_0 g(t)$ we arrive at inequality (9) in the case of negative r:

$$\left|x(t)^r - y(t)^r\right| \leq \frac{|x(t) - y(t)|}{\sum_{k=0}^{l-1} x(t)^{(l-k)/l} y(t)^{(k-rl)/l}} \leq \frac{|x(t) - y(t)|}{l(m_0 g(t))^{1-r}}.$$

In his papers Bielecki introduced for Volterra integral equations a family of norms equivalent to the supremum norm. Changing the norm of the considered function space, we are able to rewrite the Volterra integral equation as a fixed point condition of certain contractive mapping. Following this method, we construct on the space of positive continuous and bounded below functions, two families of norms determined by a scaling parameter and a non-negative function connected to the problem.

Definition 1.6

We introduce two new norms on function space $C_{m_0,g}[0,b]$

$$\|x\|_{\kappa,+} = \sup_{t\in[0,b]} |x(t)| e^{\kappa G(t)} \tag{12}$$

$$\|x\|_{\kappa,-} = \sup_{t\in[0,b]} |x(t)| e^{-\kappa G(t)}, \tag{13}$$

where G is an arbitrary continuous, non-negative function and κ is a positive real number.

It is easy to check that both norms (12), (13) are equivalent to the supremum norm on the $C_{m_0,g}[0,b]$ space.

Property 1.7

Norms $\|\cdot\|_{\kappa,-}$ and $\|\cdot\|_{\kappa,+}$ are equivalent to the supremum norm for any $\kappa \in R_+$ and function G obeying the conditions of Definition 1.6.

Proof: the equivalence is a result of the following inequalities

$$\|x\| \cdot \inf_{t\in[0,b]} e^{-\kappa G(t)} \leq \|x\|_{\kappa,-} \leq \|x\|$$

$$\|x\| \leq \|x\|_{\kappa,+} \leq \|x\| \cdot \sup_{t\in[0,b]} e^{\kappa G(t)}$$

valid for any function $x \in C_{m_0,g}[0,b]$.

2. Bielecki method for left- and right-sided integral equation on the $C_{m_0,g}[0,b]$ space

In this section we shall consider two integral equations

$$x(t) = \int_0^t K(t,s)x(s)^r\,ds + \varphi_0(t) \tag{14}$$

$$x(t) = \int_t^b K(t,s)x(s)^r\,ds + \varphi_0(t), \tag{15}$$

where $|r|<1$, kernel K is a non-negative, continuous function determined on set $[0,b]\times[0,b]$ and function $\varphi_0 \in C_{m_0,g}[0,b]$. The first of the above equations was also discussed in [21] on the space of functions continuous and bounded.
We shall prove the existence and uniqueness of the solution to the above equations in the $C_{m_0,g}[0,b]$ space. To this aim we apply the Banach theorem on a fixed point, reformulating the integral equations as fixed point conditions for the mappings defined below

$$Tx(t) := \int_0^t K(t,s)x(s)^r\,ds + \varphi_0(t) \tag{16}$$

$$\overline{T}x(t) := \int_t^b K(t,s)x(s)^r\,ds + \varphi_0(t) \tag{17}$$

Let us note that $T: C_{m_0,g}[0,b] \to C_{m_0,g}[0,b]$ and $\overline{T}: C_{m_0,g}[0,b] \to C_{m_0,g}[0,b]$. Both mappings are contractions on the $C_{m_0,g}[0,b]$ space endowed with metrics induced by norms $\|\cdot\|_{\kappa,-}$ and $\|\cdot\|_{\kappa,+}$ respectively, when $\kappa^{-1} \in (0,1)$. This result is proved in the lemma below and in Lemma 2.3.

Lemma 2.1
Mapping T, defined in formula (16) with kernel K a continuous and non-negative function on set $[0,b]\times[0,b]$, is contractive on the ($C_{m_0,g}[0,b]$, $\|\cdot\|_{\kappa,-}$) space when $\kappa^{-1} \in (0,1)$, $\varphi_0 \in C_{m_0,g}[0,b]$ and

$$G(t) = \int_0^t K(u)L(u)du \tag{18}$$

$$K(u) = \sup_{t\in[0,b]} K(t,u) \qquad\qquad L(u) = \frac{1}{l(m_0 g(u))^{1-r}}.$$

Proof: Let $x, y \in C_{m_0,g}[0,b]$ be a pair of arbitrary continuous functions bounded below by positive function $m_0 g$. The distance between their images, measured using the metric induced by norm (12) with G given in (18), fulfills the following inequalities

$$\|Tx - Ty\|_{\kappa,-} = \sup_{t\in[0,b]} e^{-\kappa G(t)} \left| \int_0^t K(t,s)x(s)^r ds - \int_0^t K(t,s)y(s)^r ds \right| \le$$

$$\le \sup_{t\in[0,b]} e^{-\kappa G(t)} \int_0^t \left| K(t,s)\left(x(s)^r - y(s)^r\right) \right| ds \le$$

$$\le \sup_{t\in[0,b]} e^{-\kappa G(t)} \int_0^t K(s)L(s)|x(s) - y(s)| ds =$$

$$= \sup_{t\in[0,b]} e^{-\kappa G(t)} \int_0^t K(s)L(s)|x(s) - y(s)| e^{-\kappa G(s)} e^{\kappa G(s)} ds \le$$

$$\le \frac{1}{\kappa} \|x - y\|_{\kappa,-} \cdot \sup_{t\in[0,b]} e^{-\kappa G(t)} \int_0^t \kappa K(s)L(s)e^{\kappa G(s)} ds =$$

$$= \frac{1}{\kappa} \|x - y\|_{\kappa,-} \cdot \sup_{t\in[0,b]} e^{-\kappa G(t)} \left(e^{\kappa G(t)} - 1\right) \le \frac{1}{\kappa} \|x - y\|_{\kappa,-} .$$

Thus, we conclude that mapping T obeys the following condition for any pair of functions $x, y \in C_{m_0,g}[0,b]$

$$\|Tx - Ty\|_{\kappa,-} \le \frac{1}{\kappa} \|x - y\|_{\kappa,-} \tag{19}$$

and for any positive value of parameter κ. Assuming $\kappa^{-1} \in (0,1)$, we note that this mapping is a contraction on the ($C_{m_0,g}[0,b]$, $\|\cdot\|_{\kappa,-}$) space.

Corollary 2.2
Equation

$$x(t) = \int_0^t K(t,s)x(s)^r ds + \varphi_0(t),$$

where $|r| < 1$ and kernel K is a non-negative, continuous function determined on set $[0,b]\times[0,b]$, has a unique solution in the $C_{m_0,g}[0,b]$ space, provided function $\varphi_0 \in C_{m_0,g}[0,b]$.

Lemma 2.3
Mapping $\overline{T}$, defined by formula (17) with kernel K a non-negative, continuous function on set $[0,b]\times[0,b]$, is contractive on the ($C_{m_0,g}[0,b]$, $\|\cdot\|_{\kappa,+}$) space when $\kappa^{-1}\in(0,1)$ and $\varphi_0\in C_{m_0,g}[0,b]$.

Proof: Let $x,y\in C_{m_0,g}[0,b]$ be a pair of arbitrary continuous functions bounded below by positive function $m_0 g$. Now the distance between their images $\overline{T}x$ and $\overline{T}y$ is measured using the metric induced by norm (13) with function *G* given in (18). It obeys the relations

$$\left\|\overline{T}x-\overline{T}y\right\|_{e,+}=\sup_{t\in[0,b]}e^{\kappa G(t)}\left|\int_t^b K(t,s)x(s)^r ds-\int_t^b K(t,s)y(s)^r ds\right|\leq$$

$$\leq\sup_{t\in[0,b]}e^{\kappa G(t)}\int_t^b\left|K(t,s)\left(x(s)^r-y(s)^r\right)\right|ds\leq\sup_{t\in[0,b]}e^{\kappa G(t)}\int_t^b K(s)L(s)\left|x(s)-y(s)\right|ds=$$

$$=\sup_{t\in[0,b]}e^{\kappa G(t)}\int_t^b K(s)L(s)\left|x(s)-y(s)\right|e^{\kappa G(s)}e^{-\kappa G(s)}ds\leq$$

$$\leq\frac{1}{\kappa}\|x-y\|_{\kappa,+}\sup_{t\in[0,b]}e^{\kappa G(t)}\int_t^b \kappa K(s)L(s)e^{-\kappa G(s)}ds=$$

$$=\frac{1}{\kappa}\|x-y\|_{\kappa,+}\sup_{t\in[0,b]}e^{\kappa G(t)}\left(e^{-\kappa G(t)}-e^{-\kappa\int_0^b K(u)L(u)du}\right)=$$

$$=\frac{1}{\kappa}\|x-y\|_{\kappa,+}\sup_{t\in[0,b]}\left(1-e^{-\kappa\int_t^b K(u)L(u)du}\right)\leq\frac{1}{\kappa}\|x-y\|_{\kappa,+}$$

The above calculations imply the following inequality valid for any functions $x,y\in C_{m_0,g}[0,b]$ and $\kappa\in R_+$

$$\left\|\overline{T}x-\overline{T}y\right\|_{\kappa,+}\leq\frac{1}{\kappa}\|x-y\|_{\kappa,+} \tag{20}$$

We now assume $\kappa^{-1}\in(0,1)$ and conclude that mapping $\overline{T}$ is a contraction in the ($C_{m_0,g}[0,b]$, $\|\cdot\|_{\kappa,+}$) space.

Corollary 2.4
Equation

$$x(t) = \int_t^b K(t,s)x(s)^r\, ds + \varphi_0(t),$$

where $|r| < 1$, kernel K is a non-negative, continuous function determined on set $[0,b]\times[0,b]$, has a unique solution in the $C_{m_0,g}[0,b]$ space, provided function $\varphi_0 \in C_{m_0,g}[0,b]$.

Let us note that the proofs of Lemma 2.1 and 2.3 imply that assumption on the continuity of kernels K can be replaced by the assumption that they are integrable and bounded functions. We give this generalized result in the following lemma describing the case of mapping T and $\overline{T}$.

Lemma 2.5
(1) Mapping T, defined in formula (16) with kernel K a non-negative function, integrable with respect to its second argument and bounded on set $[0,b]\times[0,b]$, is contractive on the $(C_{m_0,g}[0,b], \|\cdot\|_{\kappa,-})$ space when $\kappa^{-1} \in (0,1)$, $\varphi_0 \in C_{m_0,g}[0,b]$ and G is given in (18).
(2) Mapping $\overline{T}$, defined in formula (17) with kernel K a non-negative function, integrable with respect to its second argument and bounded on set $[0,b]\times[0,b]$, is contractive on the $(C_{m_0,g}[0,b], \|\cdot\|_{\kappa,+})$ space when $\kappa^{-1} \in (0,1)$, $\varphi_0 \in C_{m_0,g}[0,b]$ and G is given in (18).

3. Main results

We apply the existence and uniqueness results proved in the previous section for integral equations (14), (15) to investigate the existence of solution to fractional differential equations (7), (8). The first step is the transformation of these equations to the corresponding fixed point conditions. From the composition rules in Property 1.3, it follows that on the $C_{m_0,g}[0,b]$ space equations (7), (8) are respectively equivalent to the fractional integral equations given below

$$x(t) = \lambda \cdot I_{0+}^{\alpha} x(t)^r + \varphi_o(t) \tag{21}$$

$$x(t) = \lambda \cdot I_{b-}^{\alpha} x(t)^r + \varphi_o(t), \tag{22}$$

where functions φ_0 are stationary functions of the respective Caputo derivatives taken from the $C_{m_0,g}[0,b]$ space. Applying Definition 1.1 of the left- and right

-sided fractional integrals we conclude that the above equations are identical to earlier considered equations (14), (15), where the kernel is given by formula

$$K(t,s)=\begin{cases}\frac{\lambda}{\Gamma(\alpha)}(t-s)^{\alpha-1} & s\le t\\ 0 & s>t\end{cases} \tag{23}$$

for equation (21), $(t,s)\in[0,b]\times[0,b]$ and respectively as

$$K(t,s)=\begin{cases}0 & s<t\\ \frac{\lambda}{\Gamma(\alpha)}(s-t)^{\alpha-1} & s\ge t\end{cases} \tag{24}$$

for equation (22), $(t,s)\in[0,b]\times[0,b]$. Let us note that they are non-negative, continuous functions on set $[0,b]\times[0,b]$ when $\alpha\ge 1$ and $\lambda\ge 0$.

Proposition 3.1
If $\alpha\ge 1$, $\lambda\ge 0$ and $r\in(-1,1)$, then each stationary function φ_0 of the left-sided Caputo derivative, fulfilling the conditions: ${}^{c}D^{\alpha}_{0+}\varphi_0(t)=0$ and $\varphi_0\in C_{m_0,g}[0,b]$, generates a unique $C_{m_0,g}[0,b]$ solution of fractional differential equation

$${}^{c}D^{\alpha}_{0+}x(t)=\lambda\cdot x(t)^r\,.$$

This solution is a limit of iterations of mapping T defined below defined below on the $C_{m_0,g}[0,b]$ space:

$$Ty(t):=\lambda\cdot I^{\alpha}_{0+}y(t)^r+\varphi_o(t) \qquad y\in C_{m_0,g}[0,b]$$

$$x=\lim_{k\to\infty}(T)^k\psi\,,$$

where $\psi\in C_{m_0,g}[0,b]$ arbitrary.

Proof: as we have observed, the above fractional differential equation on space $C_{m_0,g}[0,b]$ is equivalent to equation (14) with the kernel described in (23). In addition each stationary function $\varphi_0\in C_{m_0,g}[0,b]$ creates mapping T given in (16). Thus, the assumptions of Lemma 2.1 and Corollary 2.2 are fulfilled for each stationary function of the Caputo derivative from the $C_{m_0,g}[0,b]$ space. Applying Corollary 2.2, we conclude that the considered FDE has a unique solution in this space generated by φ_0. According to the Banach theorem it is a limit of the iterations of mapping (16) with kernel (23).

In the case of equation (15) we have an analogous result which is given below. As it is a straightforward corollary of Lemma 2.3 and Corollary 2.4 we omit the proof.

Proposition 3.2

If $\alpha \geq 1$, $\lambda \geq 0$ and $r \in (-1,1)$, then each stationary function φ_0 of the right-sided Caputo derivative, fulfilling the conditions: ${}^{c}D^{\alpha}_{b-}\varphi_0(t) = 0$ and $\varphi_0 \in C_{m_0,g}[0,b]$, generates a unique $C_{m_0,g}[0,b]$ solution of fractional differential equation

$$ {}^{c}D^{\alpha}_{b-}x(t) = \lambda \cdot x(t)^r . $$

This solution is a limit of iterations of mapping *T* defined below on the $C_{m_0,g}[0,b]$ space:

$$ \bar{T}y(t) := \lambda \cdot I^{\alpha}_{b-}y(t)^r + \varphi_o(t) \qquad y \in C_{m_0,g}[0,b] $$

$$ x = \lim_{k\to\infty}(\bar{T})^k \psi , $$

where $\psi \in C_{m_0,g}[0,b]$ arbitrary.

Final remarks

In the paper we studied two types of nonlinear, one-term FDE in an arbitrary finite interval. We derived their solutions in the space of functions continuous, positive and bounded below by given function $m_0 g$. To this aim we applied the extended version of the Bielecki method and explicitly constructed solutions generated by the stationary functions of the left- and right-sided Caputo derivative.

Let us note that the scaling of norms using exponential functions, as in formulas (12), (13), restricts our results to equations of fractional order $\alpha \geq 1$. In further investigations we shall consider the case $0<\alpha<1$ using scaling via the Mittag--Leffler function which is a generalization of the exponential function.

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