APPLICATION OF THE BEM USING DISCRETIZATION IN TIME FOR NUMERICAL SOLUTION OF 3D PARABOLIC EQUATION

Grażyna Kałuża

Silesian University of Technology, Gliwice, Poland
grazyna.kaluza@polsl.pl,

Abstract. The 3D parabolic equation supplemented by adequate boundary and initial conditions is considered. This equation is solved using the combined variant of the boundary element method. The numerical model for constant boundary elements and constant internal cells is presented. In the final part of the paper the examples of computations are shown.

1. Formulation of the problem

The three-dimensional Fourier-Kirchhoff type equation is considered

\[ x \in \Omega: \frac{\partial T(x, t)}{\partial t} = a \nabla^2 T(x, t) - u \frac{\partial T(x, t)}{\partial x_i} + \frac{Q(x, t)}{c} \quad (1) \]

where \( T \) is the temperature, \( a = \lambda / c \) is the thermal diffusivity (\( \lambda \) is the thermal conductivity and \( c \) is the volumetric specific heat, respectively), \( u \) is the constant velocity, \( t \) denotes time, \( x = \{x_1, x_2, x_3\} \) and

\[ \nabla^2 T(x, t) = \frac{\partial^2 T(x, t)}{\partial x_1^2} + \frac{\partial^2 T(x, t)}{\partial x_2^2} + \frac{\partial^2 T(x, t)}{\partial x_3^2} \quad (2) \]

The equation (1) is supplemented by the boundary conditions

\[ x \in \Gamma_1: \quad T(x, t) = T_b \]
\[ x \in \Gamma_2: \quad q(x, t) = -\lambda \frac{\partial T}{\partial n} = a[T(x, t) - T_s] \quad (3) \]

and the initial one

\[ t = 0: \quad T(x, 0) = T_p \quad (4) \]
where $T_b$ is the known boundary temperature, $\alpha$ and $T_a$ are the heat transfer coefficient and the ambient temperature, respectively, $T_p$ is the initial temperature, $\partial T/\partial n$ is the normal derivative

$$\frac{\partial T}{\partial n} = \frac{\partial T}{\partial x_i} \cos \alpha_i + \frac{\partial T}{\partial x_2} \cos \alpha_2 + \frac{\partial T}{\partial x_3} \cos \alpha_3$$  \hspace{1cm} (5)$$

where $\cos \alpha_1$, $\cos \alpha_2$, $\cos \alpha_3$ are the directional cosines of the normal outward vector $n$ \[1, 2\].

\[2. \text{ Boundary element method} \]

The equation (1) can be written in the form

$$\frac{1}{a} \frac{\partial T(x, t)}{\partial t} = \nabla^2 T(x, t) - \frac{u}{a} \frac{\partial T(x, t)}{\partial x_i} + \frac{Q(x, t)}{\lambda}$$  \hspace{1cm} (6)$$

To solve the equation (6), the BEM using discretization in time is applied. At first, the following approximation with respect to time is proposed

$$\frac{1}{a} \frac{T(x, t') - T(x, t'^{-})}{\Delta t} = \nabla^2 T(x, t') - \frac{u}{a} \frac{\partial T(x, t')}{\partial x_i} + \frac{Q(x, t')}{\lambda}$$  \hspace{1cm} (7)$$

this means

$$\nabla^2 T(x, t') - \frac{1}{a \Delta t} T(x, t') - \frac{u}{a} \frac{\partial T(x, t')}{\partial x_i} + \frac{1}{a \Delta t} T(x, t'^{-}) + \frac{Q(x, t'^{-})}{\lambda} = 0$$  \hspace{1cm} (8)$$

The weighted residual criterion for equation (8) is formulated \[1, 3\]

$$\int_{\Omega} \left[ \nabla^2 T(x, t') - \frac{1}{a \Delta t} T(x, t') - \frac{u}{a} \frac{\partial T(x, t')}{\partial x_i} + \frac{1}{a \Delta t} T(x, t'^{-}) + \frac{Q(x, t'^{-})}{\lambda} \right] T^*(\xi, x) d\Omega = 0$$  \hspace{1cm} (9)$$

where $\xi$ is the observation point and $T^*(\xi, x)$ is the fundamental solution.
Using the second Green formula \[1, 4\] one has

\[
\int_{\Omega} \nabla^2 T'(\xi, x) T(x, t') d\Omega + \int_{\Gamma} \left[ T'(\xi, x) \frac{\partial T(x, t')}{\partial n} \right] d\Gamma - T(x, t') \frac{\partial T'(\xi, x)}{\partial n} d\Omega = 0
\]

or

\[
\int_{\Omega} \left[ \nabla^2 T'(\xi, x) - \frac{1}{a \Delta t} T'(\xi, x) \right] T(x, t') d\Omega - \frac{1}{\kappa_{\alpha}} \int_{\Gamma} T'(\xi, x) q(x, t') d\Gamma + \frac{1}{\kappa_{\alpha}} \int_{\Gamma} q'(\xi, x) T(x, t') d\Gamma + \frac{u}{a \Delta t} \int_{\Omega} \frac{\partial T'(x, t')}{\partial x_i} T(x, t') d\Omega = 0
\]

where \(q(x, t') = -\lambda \partial T(x, t')/\partial n\) and \(q'(\xi, x) = -\lambda \partial T'(\xi, x)/\partial n\).

Finally

\[
\int_{\Omega} \left[ \nabla^2 T'(\xi, x) - \frac{1}{a \Delta t} T'(\xi, x) + \frac{u}{a} \frac{\partial T'(x, t')}{\partial x_i} \right] T(x, t') d\Omega \]

\[
\frac{1}{\kappa_{\alpha}} \int_{\Gamma} T'(\xi, x) q(x, t') d\Gamma + \frac{1}{\kappa_{\alpha}} \int_{\Gamma} q'(\xi, x) T(x, t') d\Gamma \]

\[
\frac{u}{a \Delta t} \int_{\Gamma} T'(\xi, x) T(x, t') \cos \alpha_i d\Gamma + \frac{1}{a \Delta t} \int_{\Omega} T(x, t') T'(\xi, x) d\Omega + \frac{1}{\kappa_{\alpha}} \int_{\Gamma} Q(x, t'^{-1}) T'(\xi, x) d\Omega = 0
\]

Fundamental solution should fulfill the following equation

\[
\nabla^2 T'(\xi, x) - \frac{1}{a \Delta t} T'(\xi, x) + \frac{u}{a} \frac{\partial T'(x, t')}{\partial x_i} = -\delta(\xi, x)
\]
where $\delta(\xi, x)$ is the Dirac function.

Taking into account the property (13) the equation (12) takes a form

$$
T(\xi, t') + \frac{1}{\lambda} \int_{\Gamma} T^*(\xi, x) q(x, t') \, d\Gamma = \frac{1}{\lambda} \int_{\Gamma} q^*(\xi, x) T(x, t') \, d\Gamma - \frac{u}{a} \int_{\Gamma} T^*(\xi, x) T(x, t') \cos \alpha \, d\Gamma + \frac{1}{a \Delta t} \int_{\Omega} T^*(\xi, x) T(x, t'^{-1}) \, d\Omega + \frac{1}{\lambda} \int_{\Omega} T^*(\xi, x) \frac{T(x, t')}{d\Omega}
$$

(14)

For $\xi \in \Gamma$ the boundary integral equation is obtained

$$
B(\xi) T(\xi, t') + \frac{1}{\lambda} \int_{\Gamma} T^*(\xi, x) q(x, t') \, d\Gamma = \frac{1}{\lambda} \int_{\Gamma} q^*(\xi, x) T(x, t') \, d\Gamma - \frac{u}{a} \int_{\Gamma} T^*(\xi, x) T(x, t') \cos \alpha \, d\Gamma + \frac{1}{a \Delta t} \int_{\Omega} T^*(\xi, x) T(x, t'^{-1}) \, d\Omega + \frac{1}{\lambda} \int_{\Omega} T^*(\xi, x) \frac{T(x, t')}{d\Omega}
$$

(15)

where $B(\xi) \in (0, 1)$ is the coefficient connected with the location of point $\xi$ on the boundary $\Gamma$.

For the problem considered the fundamental solution is the following [5]

$$
T^*(\xi, x) = \frac{1}{4\pi r} \exp \left[ - \sqrt{ \left( \frac{u}{2a} \right)^2 + \frac{1}{a \Delta t} r^2 } - \frac{u}{2a} (x_i - \xi_i) \right]
$$

(16)

where $r$ is the distance between the points $\xi = (\xi_1, \xi_2, \xi_3)$ and $x = (x_1, x_2, x_3)$.

Using formula (16) the heat flux $q^*(\xi, x)$ resulting from fundamental solution can be calculated.

3. Numerical realization

To solve equation (15) the boundary is divided into $N$ boundary elements and the interior is divided into $L$ internals cells. Next, the integrals appearing in (15) are substituted by the sums of integrals [4].
So, for optional boundary point \( \xi^i \in \Gamma \) one has

\[
B(\xi^i) T(\xi^i, t^f) + \frac{1}{\lambda} \sum_{j=1}^{N} T^* (\xi^i, x) q(x, t^f) \, d\Gamma_j =
\]

\[
\frac{1}{\lambda} \sum_{j=1}^{N} \int q^* (\xi^i, x) T(x, t^f) \, d\Gamma_j - \frac{u}{a} \sum_{j=1}^{N} \int T^* (\xi^i, x) T(x, t^f) \cos \alpha_i \, d\Gamma_j + \frac{1}{\lambda} \sum_{j=1}^{N} \int T^* (\xi^i, x) \mathcal{Q}(x', t^{f-1}) \, d\Omega_j
\]

(17)

When the constant boundary elements and constant internal cells are used then the equation (17) takes form

\[
\frac{1}{2} T(\xi^i, t^f) + \frac{1}{\lambda} \sum_{j=1}^{N} q^f \int T^* (\xi^i, x) \, d\Gamma_j =
\]

\[
\frac{1}{\lambda} \sum_{j=1}^{N} T^f_j \int q^* (\xi^i, x) \, d\Gamma_j - \frac{u}{a} \sum_{j=1}^{N} T^f_j \int T^* (\xi^i, x) \cos \alpha_i \, d\Gamma_j + \frac{1}{a \Delta t} \sum_{j=1}^{L} T^f_{i-1} \int T^* (\xi^i, x) \, d\Omega_j + \frac{1}{\lambda} \sum_{j=1}^{L} Q^f_{i-1} \int T^* (\xi^i, x) \, d\Omega_i
\]

(18)

or

\[
\sum_{j=1}^{N} G_{ij} q^f_j = \sum_{j=1}^{N} (H_{ij} - U_{ij}) T^f_j + \sum_{j=1}^{L} P_{ij} T^f_{i-1} + \sum_{j=1}^{L} Z_{ij} Q^f_{i-1}, \quad i, 1, 2, \ldots, N
\]

(19)

where

\[
G_{ij} = \frac{1}{\lambda} \int T^* (\xi^i, x) \, d\Gamma_j =
\]

\[
\frac{1}{4 \pi \lambda} \int_{\xi_j}^{r_j} \left[ -\sqrt{\left( \frac{u}{2a} \right)^2 + \frac{1}{a \Delta t} r_j - \frac{u}{2a} (x_i - \xi_j)} \right] \, d\Gamma_j
\]

(20)

\[
H_{ij} = \begin{cases} 
\hat{H}_{ij}, & i \neq j \\
-\frac{1}{2}, & i = j 
\end{cases}
\]

(21)
The system of equations (19) allows one to determine the “missing” boundary values \( T_j \) and \( q_j \). Next, the temperatures at the internal points \( \xi_i, \ i = N + 1, N + 2, \ldots, N + L \) can be calculated using the formula

\[
\dot{H}_{ij} = \frac{1}{\lambda} \int_{\Gamma_j} q^* (\xi', x) d\Gamma_j = \frac{1}{4\pi} \int_{\Gamma_j} \frac{1}{r_{ij}} \exp \left[ -\sqrt{\left( \frac{u}{2a} \right)^2 + \frac{1}{a\Delta t} \frac{u}{r_{ij}} - \frac{u}{2a} (x_i - \xi_i)} \right] \cos \alpha_j + 
\]

\[
\begin{align*}
&\left\{ \sqrt{\left( \frac{u}{2a} \right)^2 + \frac{1}{a\Delta t} \frac{x_i - \xi_i}{r_{ij}} + \frac{x_i - \xi_i}{r_{ij}^2}} \cos \alpha_1 + \\
&\sqrt{\left( \frac{u}{2a} \right)^2 + \frac{1}{a\Delta t} \frac{x_i - \xi_i}{r_{ij}} + \frac{x_i - \xi_i}{r_{ij}^2}} \cos \alpha_2 + \\
&\sqrt{\left( \frac{u}{2a} \right)^2 + \frac{1}{a\Delta t} \frac{x_i - \xi_i}{r_{ij}} + \frac{x_i - \xi_i}{r_{ij}^2}} \cos \alpha_3 \right\} d\Gamma_j 
\end{align*}
\]

\[
U_{ij} = \frac{u}{a} \int_{\Gamma_j} T^* (\xi', x) \cos \alpha_i d\Gamma_j = 
\]

\[
\frac{u}{4\pi a} \int_{\Gamma_j} \frac{1}{r_{ij}} \exp \left[ -\sqrt{\left( \frac{u}{2a} \right)^2 + \frac{1}{a\Delta t} \frac{u}{r_{ij}} - \frac{u}{2a} (x_i - \xi_i)} \right] \cos \alpha_i d\Gamma_j 
\]

\[
P_{ij} = \frac{1}{\Delta t} \int_{\Omega_i} T^* (\xi', x) d\Omega_i = 
\]

\[
\frac{1}{4\pi a \Delta t} \int_{\Omega_i} \frac{1}{r_{ij}} \exp \left[ -\sqrt{\left( \frac{u}{2a} \right)^2 + \frac{1}{a\Delta t} \frac{1}{r_{ij}} - \frac{u}{2a} (x_i - \xi_i)} \right] d\Omega_i 
\]

\[
Z_{ij} = \frac{1}{\lambda} \int_{\Omega_i} T^* (\xi', x) d\Omega_i = 
\]

\[
\frac{1}{4\pi \lambda} \int_{\Omega_i} \frac{1}{r_{ij}} \exp \left[ -\sqrt{\left( \frac{u}{2a} \right)^2 + \frac{1}{a\Delta t} \frac{u}{r_{ij}} - \frac{u}{2a} (x_i - \xi_i)} \right] d\Omega_i 
\]

The system of equations (19) allows one to determine the “missing” boundary values \( T_j \) and \( q_j \). Next, the temperatures at the internal points \( \xi_i, \ i = N + 1, N + 2, \ldots, N + L \) can be calculated using the formula
\[
T_l^f = \sum_{j=1}^{N} (H_{lj} - U_{lj})T_j^f - \sum_{j=1}^{N} G_{lj} q_j^f + \sum_{l=1}^{L} P_{ll} T_l^{f-1} + \sum_{l=1}^{L} Z_{ll} Q_l^{f-1}
\]  

(26)

After determining the integrals \(G_{ij}, H_{ij}, U_{ij}, P_{il}, Z_{il}\), and taking into account the boundary conditions (3), the system of equations (19) can be solved by means of the Gaussian elimination method [4].

4. Results of computations

The following input data are introduced: thermal conductivity \(\lambda = 10 \text{ W/(mK)}\), volumetric specific heat \(c = 10^6 \text{ J/(m}^3\text{ K)}\), velocity \(u = 0.0001 \text{ m/s}\), source function \(Q = 0 \text{ W/m}^3\), initial temperature \(T_p = 0^\circ\text{C}\) and the time step \(\Delta t = 5 \text{ s}\).

The cuboid of dimensions \(l_1 \times l_2 \times l_3 = 0.05 \times 0.05 \times 0.025 \text{ m}^3\) is considered. It’s assumed that \(n_1 = n_2 = 10, n_3 = 5\), so \(N = 400\) boundary elements have been distinguished.
In the example 1 at the left and right surfaces of cuboid the temperatures 50°C and 100°C, respectively have been assumed, on the remaining boundaries the no-heat flux condition has been accepted. In Figure 1 the temperature distribution in the plane $x_2 = 0.0225$ m is shown. In the example 2 at the right surface the Robin condition has been assumed ($\alpha = 50$ W/(m$^2$K), $T_a = 20°C$). Results of computations are shown in Figure 2.

The boundary conditions are assumed in the form assuring the possibility of solution verification (the 3D problem becomes practically the 1D one). The analytical solution concerning the steady state is very simple and it can be compared with the numerical one at time for which the stabilization of temperature field takes place. Additionally the paralell position and linear shape of isotherms should be observed.

Acknowledgement

This paper is a part of Grant No N N501 3667 34.

References