APPLICATION OF THE BEM USING DISCRETIZATION IN TIME FOR NUMERICAL SOLUTION OF 3D PARABOLIC EQUATION

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Abstract. The 3D parabolic equation supplemented by adequate boundary and initial conditions is considered. This equation is solved using the combined variant of the boundary element method. The numerical model for constant boundary elements and constant internal cells is presented. In the final part of the paper the examples of computations are shown.

1. Formulation of the problem

The three-dimensional Fourier-Kirchhoff type equation is considered

$$x \in \Omega: \quad \frac{\partial T(x,t)}{\partial t} = a\nabla^2 T(x,t) - u\,\frac{\partial T(x,t)}{\partial x_1} + \frac{Q(x,t)}{c} \tag{1}$$

where T is the temperature, $a = \lambda/c$ is the thermal diffusivity (λ is the thermal conductivity and c is the volumetric specific heat, respectively), u is the constant velocity, t denotes time, $x = \{x_1, x_2, x_3\}$ and

$$\nabla^2 T(x,t) = \frac{\partial^2 T(x,t)}{\partial x_1^2} + \frac{\partial^2 T(x,t)}{\partial x_2^2} + \frac{\partial^2 T(x,t)}{\partial x_3^2} \tag{2}$$

The equation (1) is supplemented by the boundary conditions

$$\begin{aligned} x \in \Gamma_1 &: \quad T(x,t) = T_b \\ x \in \Gamma_2 &: \quad q(x,t) = -\lambda\,\frac{\partial T}{\partial n} = \alpha\left[T(x,t) - T_a\right] \end{aligned} \tag{3}$$

and the initial one

$$t = 0: \quad T(x,0) = T_p \tag{4}$$

where T_b is the known boundary temperature, α and T_a are the heat transfer coefficient and the ambient temperature, respectively, T_p is the initial temperature, $\partial T/\partial n$ is the normal derivative

$$\frac{\partial T}{\partial n} = \frac{\partial T}{\partial x_1}\cos\alpha_1 + \frac{\partial T}{\partial x_2}\cos\alpha_2 + \frac{\partial T}{\partial x_3}\cos\alpha_3 \tag{5}$$

where $\cos\alpha_1$, $\cos\alpha_2$, $\cos\alpha_3$ are the directional cosines of the normal outward vector n [1, 2].

2. Boundary element method

The equation (1) can be written in the form

$$\frac{1}{a}\frac{\partial T(x, t)}{\partial t} = \nabla^2 T(x, t) - \frac{u}{a}\frac{\partial T(x, t)}{\partial x_1} + \frac{Q(x, t)}{\lambda} \tag{6}$$

To solve the equation (6), the BEM using discretization in time is applied. At first, the following approximation with respect to time is proposed

$$\frac{1}{a}\frac{T(x, t^f) - T(x, t^{f-1})}{\Delta t} = \nabla^2 T(x, t^f) - \frac{u}{a}\frac{\partial T(x, t^f)}{\partial x_1} + \frac{Q(x, t^f)}{\lambda} \tag{7}$$

this means

$$\nabla^2 T(x, t^f) - \frac{1}{a\,\Delta t}T(x, t^f) - \frac{u}{a}\frac{\partial T(x, t^f)}{\partial x_1} + \frac{1}{a\,\Delta t}T(x, t^{f-1}) + \frac{Q(x, t^{f-1})}{\lambda} = 0 \tag{8}$$

The weighted residual criterion for equation (8) is formulated [1, 3]

$$\int_\Omega \left[\nabla^2 T(x, t^f) - \frac{1}{a\,\Delta t}T(x, t^f) - \frac{u}{a}\frac{\partial T(x, t^f)}{\partial x_1} + \frac{1}{a\,\Delta t}T(x, t^{f-1}) + \frac{Q(x, t^{f-1})}{\lambda}\right] T^*(\xi, x)\,\mathrm{d}\Omega = 0 \tag{9}$$

where ξ is the observation point and $T^*(\xi, x)$ is the fundamental solution.

Using the second Green formula [1, 4] one has

$$\int_\Omega \nabla^2 T^*(\xi, x)\, T(x, t^f)\,\mathrm{d}\Omega + \int_\Gamma \left[T^*(\xi, x)\, \frac{\partial T(x, t^f)}{\partial n} - T(x, t^f)\, \frac{\partial T^*(\xi, x)}{\partial n} \right] \mathrm{d}\Gamma - \frac{1}{a\,\Delta t} \int_\Omega T(x, t^f)\, T^*(\xi, x)\,\mathrm{d}\Omega - \frac{u}{a}\int_\Omega \frac{\partial T(x,\ t^f)}{\partial x_1}\, T^*(\xi, x)\,\mathrm{d}\Omega + \frac{1}{a\,\Delta t}\int_\Omega T(x, t^{f-1})\, T^*(\xi, x)\,\mathrm{d}\Omega + \frac{1}{\lambda}\int_\Omega Q(x, t^{f-1})\, T^*(\xi, x)\,\mathrm{d}\Omega = 0 \tag{10}$$

or

$$\int_\Omega \left[\nabla^2 T^*(\xi, x) - \frac{1}{a\,\Delta t}\, T^*(\xi, x) \right] T(x, t^f)\,\mathrm{d}\Omega - \frac{1}{\lambda}\int_\Gamma T^*(\xi, x)\, q(x, t^f)\,\mathrm{d}\Gamma + \frac{1}{\lambda}\int_\Gamma q^*(\xi, x)\, T(x, t^f)\,\mathrm{d}\Gamma + \frac{u}{a}\int_\Omega \frac{\partial T^*(x, t^f)}{\partial x_1}\, T(x, t^f)\,\mathrm{d}\Omega - \frac{u}{a}\int_\Gamma T^*(\xi, x)\, T(x, t^f) \cos\alpha_1\,\mathrm{d}\Gamma + \frac{1}{a\,\Delta t}\int_\Omega T(x, t^{f-1})\, T^*(\xi, x)\,\mathrm{d}\Omega + \frac{1}{\lambda}\int_\Omega Q(x, t^{f-1})\, T^*(\xi, x)\,\mathrm{d}\Omega = 0 \tag{11}$$

where $q(x, t^f) = -\lambda\, \partial T(x, t^f)/\partial n$ and $q^*(\xi, x) = -\lambda\, \partial T^*(\xi, x)/\partial n$.
Finally

$$\int_\Omega \left[\nabla^2 T^*(\xi, x) - \frac{1}{a\,\Delta t}\, T^*(\xi, x) + \frac{u}{a}\, \frac{\partial T^*(x, t^f)}{\partial x_1} \right] T(x, t^f)\,\mathrm{d}\Omega - \frac{1}{\lambda}\int_\Gamma T^*(\xi, x)\, q(x, t^f)\,\mathrm{d}\Gamma + \frac{1}{\lambda}\int_\Gamma q^*(\xi, x)\, T(x, t^f)\,\mathrm{d}\Gamma - \frac{u}{a}\int_\Gamma T^*(\xi, x)\, T(x, t^f)\cos\alpha_1\,\mathrm{d}\Gamma + \frac{1}{a\,\Delta t}\int_\Omega T(x, t^{f-1})\, T^*(\xi, x)\,\mathrm{d}\Omega + \frac{1}{\lambda}\int_\Omega Q(x, t^{f-1})\, T^*(\xi, x)\,\mathrm{d}\Omega = 0 \tag{12}$$

Fundamental solution should fulfill the following equation

$$\nabla^2 T^*(\xi, x) - \frac{1}{a\,\Delta t}\, T^*(\xi, x) + \frac{u}{a}\, \frac{\partial T^*(x, t^f)}{\partial x_1} = -\delta(\xi, x) \tag{13}$$

where $\delta(\xi, x)$ is the Dirac function.

Taking into account the property (13) the equation (12) takes a form

$$T(\xi, t^f) + \frac{1}{\lambda}\int_\Gamma T^*(\xi, x)\, q(x, t^f)\,\mathrm{d}\Gamma = \frac{1}{\lambda}\int_\Gamma q^*(\xi, x)\, T(x, t^f)\,\mathrm{d}\Gamma - \frac{u}{a}\int_\Gamma T^*(\xi, x)\, T(x, t^f)\cos\alpha_1\,\mathrm{d}\Gamma + \frac{1}{a\,\Delta t}\int_\Omega T^*(\xi, x) T(x, t^{f-1})\,\mathrm{d}\Omega + \frac{1}{\lambda}\int_\Omega T^*(\xi, x)\, Q(x, t^{f-1})\,\mathrm{d}\Omega \tag{14}$$

For $\xi \in \Gamma$ the boundary integral equation is obtained

$$B(\xi)\, T(\xi, t^f) + \frac{1}{\lambda}\int_\Gamma T^*(\xi, x)\, q(x, t^f)\,\mathrm{d}\Gamma = \frac{1}{\lambda}\int_\Gamma q^*(\xi, x)\, T(x, t^f)\,\mathrm{d}\Gamma - \frac{u}{a}\int_\Gamma T^*(\xi, x)\, T(x, t^f)\cos\alpha_1\,\mathrm{d}\Gamma + \frac{1}{a\,\Delta t}\int_\Omega T^*(\xi, x) T(x, t^{f-1})\,\mathrm{d}\Omega + \frac{1}{\lambda}\int_\Omega T^*(\xi, x)\, Q(x, t^{f-1})\,\mathrm{d}\Omega \tag{15}$$

where $B(\xi) \in (0, 1)$ is the coefficient connected with the location of point ξ on the boundary Γ.

For the problem considered the fundamental solution is the following [5]

$$T^*(\xi, x) = \frac{1}{4\pi r}\exp\left[-\sqrt{\left(\frac{u}{2a}\right)^2 + \frac{1}{a\,\Delta t}}\; r - \frac{u}{2a}(x_1 - \xi_1)\right] \tag{16}$$

where r is the distance between the points $\xi = (\xi_1, \xi_2, \xi_3)$ and $x = (x_1, x_2, x_3)$.

Using formula (16) the heat flux $q^*(\xi, x)$ resulting from fundamental solution can be calculated.

3. Numerical realization

To solve equation (15) the boundary is divided into N boundary elements and the interior is divided into L internals cells. Next, the integrals appearing in (15) are substituted by the sums of integrals [4].

So, for optional boundary point $\xi^i \in \Gamma$ one has

$$B\left(\xi^i\right) T(\xi^i, t^f) + \frac{1}{\lambda}\sum_{j=1}^{N}\int_{\Gamma_j} T^*\left(\xi^i, x\right) q(x, t^f)\,\mathrm{d}\Gamma_j =$$

$$\frac{1}{\lambda}\sum_{j=1}^{N}\int_{\Gamma_j} q^*\left(\xi^i, x\right) T(x, t^f)\,\mathrm{d}\Gamma_j - \frac{u}{a}\sum_{j=1}^{N}\int_{\Gamma_j} T^*\left(\xi^i, x\right) T(x, t^f)\cos\alpha_1 \mathrm{d}\Gamma_j \tag{17}$$

$$+ \frac{1}{a\,\Delta t}\sum_{l=1}^{L}\int_{\Omega_l} T^*\left(\xi^i, x\right) T(x^l, t^{f-1})\,\mathrm{d}\Omega_l + \frac{1}{\lambda}\sum_{l=1}^{L}\int_{\Omega_l} T^*\left(\xi^i, x\right) Q(x^l, t^{f-1})\,\mathrm{d}\Omega_l$$

When the constant boundary elements and constant internal cells are used then the equation (17) takes form

$$\frac{1}{2} T(\xi^i, t^f) + \frac{1}{\lambda}\sum_{j=1}^{N} q_j^f \int_{\Gamma_j} T^*\left(\xi^i, x\right)\mathrm{d}\Gamma_j =$$

$$\frac{1}{\lambda}\sum_{j=1}^{N} T_j^f \int_{\Gamma_j} q^*\left(\xi^i, x\right)\mathrm{d}\Gamma_j - \frac{u}{a}\sum_{j=1}^{N} T_j^f \int_{\Gamma_j} T^*\left(\xi^i, x\right)\cos\alpha_1\,\mathrm{d}\Gamma_j + \tag{18}$$

$$\frac{1}{a\,\Delta t}\sum_{l=1}^{L} T_l^{f-1}\int_{\Omega_l} T^*\left(\xi^i, x\right)\mathrm{d}\Omega_l + \frac{1}{\lambda}\sum_{l=1}^{L} Q_l^{f-1}\int_{\Omega_l} T^*\left(\xi^i, x\right)\mathrm{d}\Omega_l$$

or

$$\sum_{j=1}^{N} G_{ij} q_j^f = \sum_{j=1}^{N}\left(H_{ij} - U_{ij}\right) T_j^f + \sum_{l=1}^{L} P_{il}\, T_l^{f-1} + \sum_{l=1}^{L} Z_{il}\, Q_l^{f-1}, \quad i = 1, 2, \ldots, N \tag{19}$$

where

$$G_{ij} = \frac{1}{\lambda}\int_{\Gamma_j} T^*\left(\xi^i, x\right)\mathrm{d}\Gamma_j =$$

$$\frac{1}{4\pi\lambda}\int_{\Gamma_j}\frac{1}{r_{ij}}\exp\left[-\sqrt{\left(\frac{u}{2a}\right)^2 + \frac{1}{a\,\Delta t}}\; r_{ij} - \frac{u}{2a}\,(x_1 - \xi_1)\right]\mathrm{d}\Gamma_j \tag{20}$$

$$H_{ij} = \begin{cases} \hat{H}_{ij}, & i \neq j \\ -\dfrac{1}{2}, & i = j \end{cases} \tag{21}$$

$$\hat{H}_{ij} = \frac{1}{\lambda}\int_{\Gamma_j} q^*\left(\xi^i, x\right)\mathrm{d}\Gamma_j = \frac{1}{4\pi}\int_{\Gamma_j}\frac{1}{r_{ij}}\exp\left[-\sqrt{\left(\frac{u}{2a}\right)^2 + \frac{1}{a\Delta t}}\; r_{ij} - \frac{u}{2a}\,(x_1-\xi_1)\right]$$
$$\left\{\left[\sqrt{\left(\frac{u}{2a}\right)^2+\frac{1}{a\Delta t}}\;\frac{x_1-\xi_1}{r_{ij}} + \frac{x_1-\xi_1}{r_{ij}^{\;2}} + \frac{u}{2a}\right]\cos\alpha_1 + \right. \tag{22}$$
$$\left[\sqrt{\left(\frac{u}{2a}\right)^2+\frac{1}{a\Delta t}}\;\frac{x_2-\xi_2}{r_{ij}} + \frac{x_2-\xi_2}{r_{ij}^{\;2}}\right]\cos\alpha_2 +$$
$$\left.\left[\sqrt{\left(\frac{u}{2a}\right)^2+\frac{1}{a\Delta t}}\;\frac{x_3-\xi_3}{r_{ij}} + \frac{x_3-\xi_3}{r_{ij}^{\;2}}\right]\cos\alpha_3\right\}\mathrm{d}\Gamma_j$$

$$U_{ij} = \frac{u}{a}\int_{\Gamma_j} T^*\left(\xi^i,\, x\right)\cos\alpha_1 \mathrm{d}\Gamma_j = \tag{23}$$
$$\frac{u}{4\pi a}\int_{\Gamma_j}\left(\frac{1}{r_{ij}}\exp\left[-\sqrt{\left(\frac{u}{2a}\right)^2+\frac{1}{a\Delta t}}\; r_{ij} - \frac{u}{2a}\,(x_1-\xi_1)\right]\right)\cos\alpha_1\,\mathrm{d}\Gamma_j$$

$$P_{il} = \frac{1}{a\Delta t}\int_{\Omega_l} T^*\left(\xi^i,\, x\right)\mathrm{d}\Omega_l = \tag{24}$$
$$\frac{1}{4\pi a\Delta t}\int_{\Omega_l}\frac{1}{r_{il}}\exp\left[-\sqrt{\left(\frac{u}{2a}\right)^2+\frac{1}{a\Delta t}}\; r_{il} - \frac{u}{2a}\,(x_1-\xi_1)\right]\mathrm{d}\Omega_l$$

$$Z_{il} = \frac{1}{\lambda}\int_{\Omega_l} T^*\left(\xi^i,\, x\right)\mathrm{d}\Omega_l = \tag{25}$$
$$\frac{1}{4\pi\lambda}\int_{\Omega_l}\frac{1}{r_{il}}\exp\left[-\sqrt{\left(\frac{u}{2a}\right)^2+\frac{1}{a\Delta t}}\; r_{il} - \frac{u}{2a}\,(x_1-\xi_1)\right]\mathrm{d}\Omega_l$$

The system of equations (19) allows one to determine the “missing” boundary values T_j and q_j. Next, the temperatures at the internal points ξ^i, $i = N + 1, N + 2, \ldots N + L$ can be calculated using the formula

$$T_i^{\,f} = \sum_{j=1}^{N}\left(H_{ij} - U_{ij}\right)T_j^{\,f} - \sum_{j=1}^{N} G_{ij}\, q_j^{\,f} + \sum_{l=1}^{L} P_{il}\, T_l^{\,f-1} + \sum_{l=1}^{L} Z_{il}\, Q_l^{\,f-1} \tag{26}$$

After determining the integrals G_{ij}, H_{ij}, U_{ij}, P_{il}, Z_{il}, and taking into account the boundary conditions (3), the system of equations (19) can be solved by means of the Gaussian elimination method [4].

4. Results of computations

The following input data are introduced: thermal conductivity λ = 10 W/(mK), volumetric specific heat c = 10^6 J/(m^3K), velocity u = 0.0001 m/s, source function Q = 0 W/m^3, initial temperature T_p = 0°C and the time step Δt = 5 s.

The cuboid of dimensions $l_1 \times l_2 \times l_3$ = 0.05 × 0.05 × 0.025 m^3 is considered. It's assumed that n_1 = n_2 = 10, n_3 = 5, so N = 400 boundary elements have been distinguished.

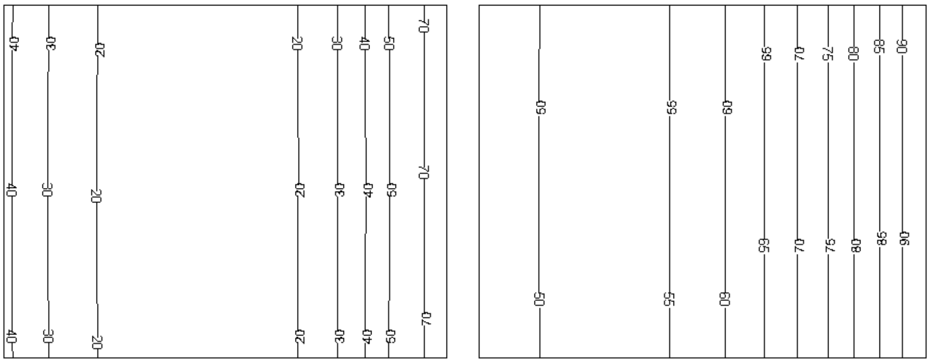

Fig. 1. Temperature distribution for times 10 and 50 second - 1 example (plane x_2 = 0.0225 m)

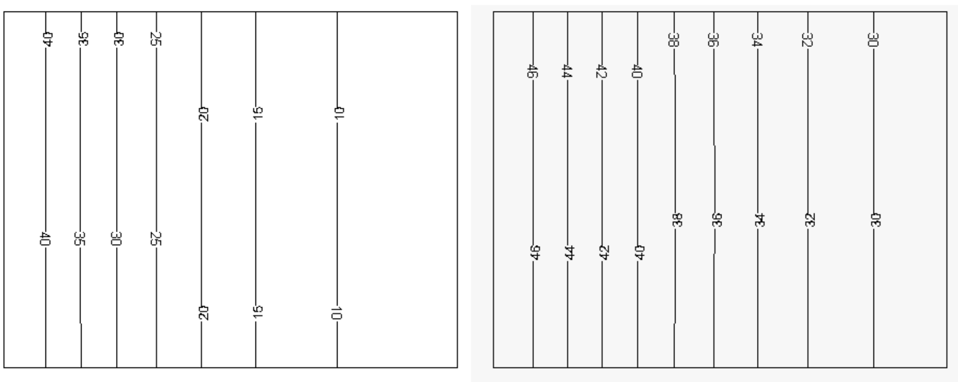

Fig. 2. Temperature distribution for times 30 and 100 second - 2 example (plane x_2 = 0.0225 m)

In the example 1 at the left and right surfaces of cuboid the temperatures 50°C and 100°C, respectively have been assumed, on the remaining boundaries the no-heat flux condition has been accepted. In Figure 1 the temperature distribution in the plane x_2 = 0.0225 m is shown. In the example 2 at the right surface the Robin condition has been assumed (α = 50 W/(m^{2}K), T_a = 20°C). Results of computations are shown in Figure 2.

The boundary conditions are assumed in the form assuring the possibility of solution verification (the 3D problem becomes practically the 1D one). The analytical solution concerning the steady state is very simple and it can be compared with the numerical one at time for which the stabilization of temperature field takes place. Additionally the paralell position and linear shape of isotherms should be observed.

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