APPLICATION OF SIMPLIFIED CURVED BOUNDARY ELEMENTS TO THE PLATE ANALYSIS - PART ONE

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Abstract. Static analysis of Kirchhoff plate by the Boundary Element Method is presented in the paper. The Bettie theorem is used to derive the boundary integral equation. Simplified curved elements are introduced. Modified approach of boundary integral equation formulation is adopted in which there is no need to introduce the equivalent shear forces at the boundary and concentrated forces at the plate corners. The collocation version of boundary element method with singular and non-singular approach is presented.

Introduction

The Boundary Element Method (BEM) was created as a completely independent numerical tool to solve engineering problems [1, 2]. The BEM do not require the all domain discretization but only the boundary of a considered structure. The Boundary Element Method is often used in the theory of both thin and thick plates and is particularly suitable to analyse the plates of arbitrary shapes and rested on internal supports. Analysis of plate bending using BEM was introduced by Bèzine [3] and Stern [4] for Kirchhoff plate theory and by Vander Weeën [5] for the thick plate theory. Okupniak and Sygulski [6] used fundamental solution of Reissner plate proposed by Ganowicz [7]. Altiero and Sikarskie [8] and Debbih [9, 10] proposed BEM to plate bending problems. Beskos [11], Wen, Aliabadi and Young [12] applied BEM to dynamic problem of plates. Some authors present a modified approach of thin plate analysis. El-Zafrany, Debbih and Fadhil [13] assumed non-zero distribution of stress over the plate thickness. Abdel-Akher and Hartley [14] worked out the method of fundamental function integration connected to external distributed loading. Guminiak, Okupniak and Sygulski [15] assumed a physical boundary condition also discussed in this paper. Present paper includes a modified formulation for bending analysis of plates, in which three geometric and three static variables at the plate boundary are considered. Application of curved boundary elements to structural analysis was proposed by Wrobel and Aliabadi [2]. Authors proposed three-node continuous and discontinuous quadratic elements also discussed in this paper. In the paper curved, simplified boundary elements are introduced into the thin plate analysis. This part of elaborations includes theoretical aspect of thin plate bending problem using the boundary element method.

1. Integral formulation of thin plate bending in modified approach

On the plate boundary, there are considered following variables: the shear force T_n, bending moment M_n, twisting moment M_{ns} and deflection w, angle of rotation in normal direction φ_n and angle of rotation in tangent direction φ_s. Only two of them are independent. The boundary integral equation is derived using Bettie theorem. Two plates are considered: infinite plate, subjected unit concentrated loading and the real one. As a result the boundary integral equation is in the form [15]:

$$\begin{aligned} & c(\mathbf{x})\cdot w(\mathbf{x}) + \int_\Gamma \left[T_n^*(\mathbf{y},\mathbf{x})\cdot w_b(\mathbf{y}) - M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y}) - M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})\right]\cdot d\Gamma(\mathbf{y}) = \\ & = \int_\Gamma \left[T_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x}) - M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x}) - M_{ns}(\mathbf{y})\cdot\varphi_s^*(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y}) + \\ & + \int_\Omega p(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})\cdot d\Omega(\mathbf{y}) \end{aligned} \tag{1}$$

where the fundamental solution of biharmonic equation $\nabla^4 w = (1/D)\cdot\bar{\delta}(\mathbf{y}-\mathbf{x})$ is given as a Green function

$$w^*(\mathbf{y},\mathbf{x}) = \frac{1}{D}\frac{r^2}{8\pi}\ln r \tag{2}$$

for a thin isotropic plate, $r = |\mathbf{y}-\mathbf{x}|$, $\bar{\delta}$ is Dirac delta and $D = \left(E\,h_{\mathrm{p}}^3\right)/\left(12\,\left(1-\nu_{\mathrm{p}}^2\right)\right)$ is a plate stiffness. The coefficient $c(\mathbf{x})$ depends on localization of point $\mathbf{x}$ and $c(\mathbf{x}) = 1$, when $\mathbf{x}$ is located inside the plate region, $c(\mathbf{x}) = 0.5$, when $\mathbf{x}$ is located on the smooth boundary and $c(\mathbf{x}) = 0$, when $\mathbf{x}$ is located outside the plate region.

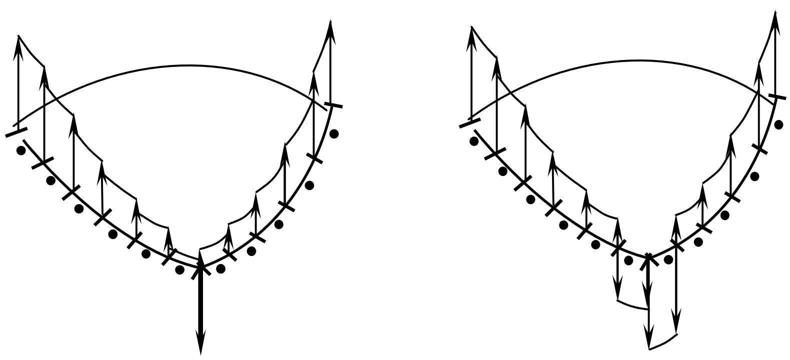

Fig. 1. Distribution of the support reaction - classic and present approach

The second equation can be derived by substituting of unit concentrated force $P^* = 1^*$ unit concentrated moment $M_n^* = 1^*$. It is equivalent to differentiate the first boundary integral equation (1) on *n* direction in point **x** on a plate boundary.

Idea of the proposed approach and formulation of boundary integral equation in plate bending is shown in Figure 1.

2. Boundary conditions

The boundary conditions for clamped edge are formulated as follows [15]:

$$w = 0, \varphi_n = 0, \varphi_s = 0, M_{ns} = 0 \tag{3}$$

The unknown variables are: the bending moment M_n and the shear force T_n.

For simply-supported edge the boundary conditions have the form [15]:

$$w = 0, \varphi_s = 0, M_n = 0, M_{ns} = 0 \tag{4}$$

The unknown values are: the shear force T_n and the angle of rotation in direction *n*, φ_n.

Free edge can be described by the following boundary conditions [15]:

$$T_n = 0, M_n = 0, M_{ns} = 0 \tag{5}$$

The unknown variables are: the deflection w and the angles of rotation φ_n, φ_s. Because the relation between φ_s and w is known, $\varphi_s = \partial w / \partial s$, there are only two independent values: w and φ_n. The parameter $\partial w(\mathbf{y}) / \partial s$ can be calculated approximately by constructing a differential expression using deflections of three neighbouring nodes.

3. Types of boundary element

Definition of curvilinear boundary element may be different. It is possible to define geometry of element considering three nodal points and only one collocation point connected with relevant physical boundary value. This type of element can be called as curved, "constant" type. The collocation point may be located slightly outside of plate edge (Fig. 2a) or exactly on the element (Fig. 2b).

Geometry of the element is defined using polynominal function, described in standard coordinate system $\langle -1, 0, 1 \rangle$. The functions are in the form:

$$N_1 = \frac{1}{2}\eta\,(1-\eta),\, N_2 = 1-\eta^2,\, N_3 = \frac{1}{2}\eta\,(1+\eta) \tag{6}$$

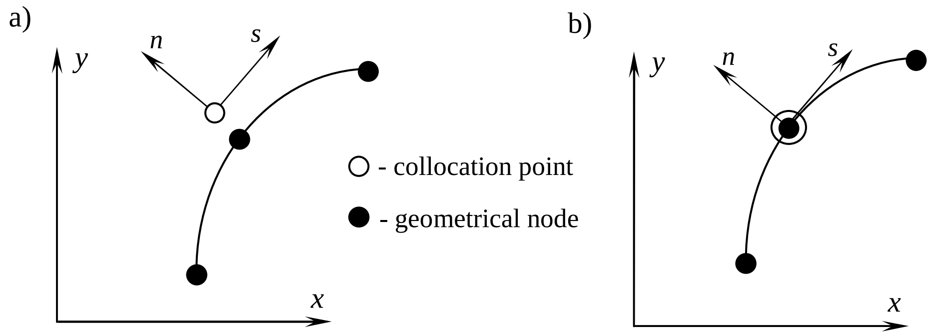

Fig. 2. Curved boundary element of constant type in non-singular and singular definition

Another type of curved element is typical isoparametric element with three geometrical nodes and three nodes connected with suitable boundary variables (Fig. 3).

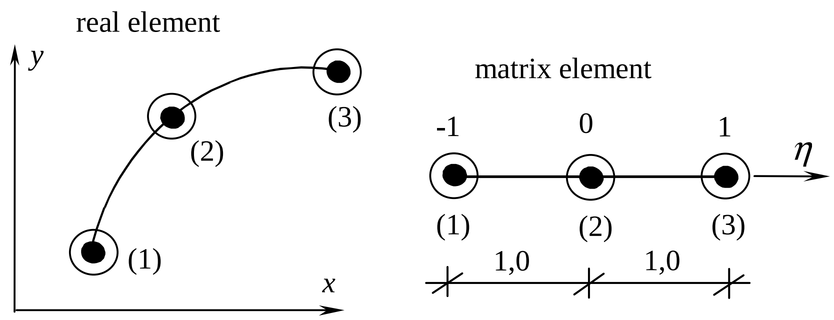

Fig. 3. Continuous, curved boundary element in singular formulation

Discontinuous boundary element is defined if localization of external physical nodes (collocation points) does not cover with geometrical nodes (Fig. 4).

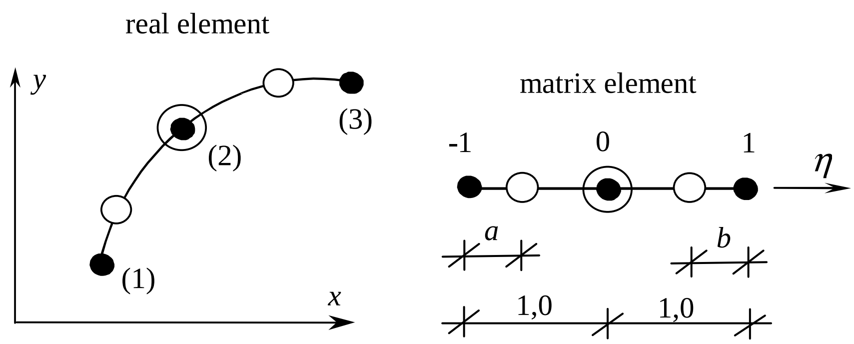

Fig. 4. Discontinuous, curved boundary element in singular formulation

The geometry of this element is defined using functions describing by equation (6). Suitable boundary variables taking a stand in boundary integral equation (1) are defined according to following functions [2]:

$$\begin{cases} \tilde{N}_1 = \dfrac{l\eta \cdot (l\eta - l + 2b)}{2 \cdot (l - a - b) \cdot (l - 2a)} \\ \tilde{N}_2 = \dfrac{l\eta \cdot [2(a-b) - l\eta]}{(l-2a) \cdot (l-2b)} + 1 \\ \tilde{N}_3 = \dfrac{l\eta \cdot (l\eta + l - 2b)}{2 \cdot (l - a - b) \cdot (l - 2b)} \end{cases} \tag{7}$$

where l is the length of the matrix element, $\eta \in \langle -1; 1 \rangle$, a is the distance between first nodes: geometrical and physical, b is the distance between last nodes: geometrical and physical (Fig. 4).

4. Construction of set of algebraic equation

A plate edge is discretized using boundary elements. In matrix notation the set of algebraic equation has the form:

$$\mathbf{G} \cdot \overline{\mathbf{B}} = \mathbf{F} \tag{8}$$

where $\mathbf{G}$ is matrix of suitable boundary integrals, $\overline{\mathbf{B}}$ is the vector on unknown variables and $\mathbf{F}$ is right-hand-side vector. If on the part of plate boundary free edge takes place, then equation (8) may be prescribed to the form:

$$\begin{bmatrix} \mathbf{G}_{\mathbf{BB}} & \mathbf{G}_{\mathbf{BS}} \\ \mathbf{\Delta} & -\mathbf{I} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{B} \\ \varphi_{\mathrm{s}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{\mathbf{B}} \\ \mathbf{0} \end{Bmatrix} \tag{9}$$

where $\mathbf{I}$ is unit matrix and $\mathbf{\Delta}$ is matrix constructed using suitable difference equation. These equation will be discussed in point 4.1. In present formulation of boundary integral equation, angles of rotation in tangent direction φ_s are additional unknown variables.

4.1. Construction of characteristic matrix

The boundary integral equation will be formulated in singular and non-singular approach. To construct the characteristic matrix $\mathbf{G}$, integration of suitable fundamental function on boundary is needed. Integration is done in local coordinate system n_i, s_i connected with i^{th} boundary element and next, these integrals must be transformed to n_k, s_k coordinate system, connected with k^{th} element (Fig. 5).

Let it be assumed, that boundary integral equation will be formulated in non-singular approach. The collocation point is located slightly outside of the plate edge. Localization of collocation point is defined by the parameter δ or non-dimensional parameter ε. This parameter can be defined as $\varepsilon = \delta/c$, where c is the length of element chord.

To calculate elements of characteristic matrix are applied following methods:

a) classic, numerical Gauss procedure for non-quasi diagonal elements,
b) modified, numerical integration of Gauss method for quasi-diagonal elements [6].

Boundary integrals on curved element are calculated according to Gauss method:

$$I = \int_a^b f^*(s) \cdot ds = \int_{-1}^{1} f^*(\eta) \cdot J(\eta) \cdot d\eta = \sum_{j=1}^{m} W_j \cdot f^*(\eta_j) \cdot J(\eta_j) \qquad j = 1, 2, 3 ... m \tag{10}$$

where f^* is suitable fundamental function, m is the number of Gauss point, W_j is the weight of j^{th} point, $J(\eta_j)$ is Jacobian of transformation, and η_j is abscissa of j^{th} point. Fundamental functions are expressed using n_i, s_i coordinate system. Then, all coordinates of Gauss points are expressed as:

$$\begin{aligned} n_i^{(\eta)} &= N_1(\eta) \cdot n_i^{(1)} + N_2(\eta) \cdot n_i^{(2)} + N_3(\eta) \cdot n_i^{(3)} \\ s_i^{(\eta)} &= N_1(\eta) \cdot s_i^{(1)} + N_2(\eta) \cdot s_i^{(2)} + N_3(\eta) \cdot s_i^{(3)} \end{aligned} \tag{11}$$

where coordinates of three geometrical nodes are used: n_i, s_i. Jacobian of transformation is expressed as:

$$J(\eta) = \sqrt{\left[\frac{dn(\eta)}{d\eta}\right]^2 + \left[\frac{ds(\eta)}{d\eta}\right]^2} \tag{12}$$

Integrals of fundamental functions are calculated using n_i, s_i coordinate system, connected with i^{th} boundary element. Then, they are transformed to n_k, s_k coordinate system on the following way:

$$\varphi_{n_k}^* = \varphi_{n_i}^* \cdot c_{nn} + \varphi_{s_i}^* \cdot c_{ns} \tag{13}$$

$$M_{n_k}^* = M_{n_i}^* \cdot c_{nn}^2 + M_{s_i}^* \cdot c_{ns}^2 + 2 \cdot M_{n_i s_i}^* \cdot c_{nn} \cdot c_{ns} \tag{14}$$

$$M_{n_k s_k}^* = \left(M_{s_i}^* - M_{n_i}^*\right) \cdot c_{nn} \cdot c_{ns} + M_{n_i s_i}^* \cdot \left[c_{nn}^2 - c_{ns}^2\right] \tag{15}$$

$$T^*_{n_k} = T^*_{n_i} \cdot c_{nn} + T^*_{s_i} \cdot c_{ns} \tag{16}$$

where $c_{nn} = \cos(n_k, n_i)$ and $c_{ns} = \cos(n_k, s_i)$.

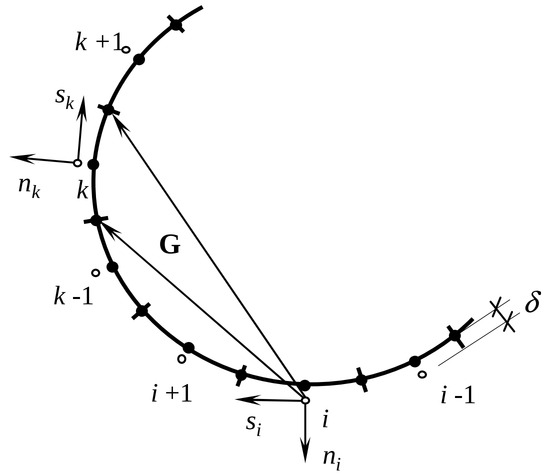

Fig. 5. Construction of characteristic matrix

To calculate quasi-diagonal integrals using non-singular approach, Gauss |method of integration also can be used. Because some of fundamental functions have extensive gradients in circumvolution of collocation point, the classic Gauss method does not give sufficiently accurate results. Hence, the manner proposed by Okupniak and Sygulski is applied [6]. Authors proposed inverse localization of Gauss points in domain of integration (Fig. 6).

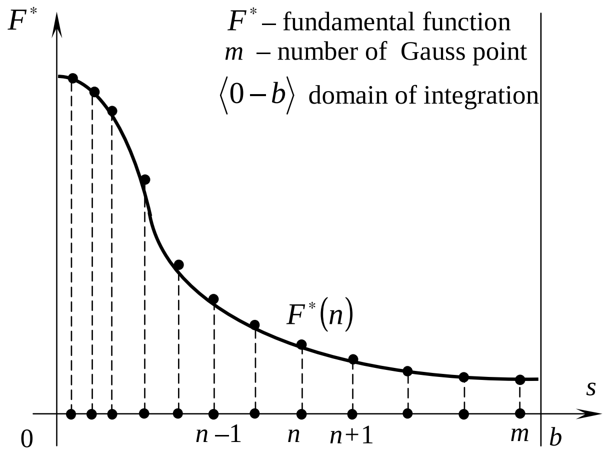

Fig. 6. Calculation of quasi-diagonal integrals using modified Gauss method

In case of a free edge, it is needed to extend of characteristic matrix to elements connected with angle of rotation in tangent direction φ_s. The characteristic matrix will have extended dimension $\overline{N} \times \overline{N}$, where: $\overline{N} = N + N_s$, and N_s is number of additional geometrical parameters φ_s. Additional boundary integrals are localized in matrix $\mathbf{G_{BS}}$ according to matrix equation (9).

In singular approach, when the collocation point is localized exactly on the boundary, the quasi-diagonal boundary integrals are calculated differently. The non quasi-diagonal integrals of characteristic matrix are calculated using standard Gauss method. The fundamental function w^* has the singularity of the second order. For the clamped edge, integrals of functions: w^*, φ_n^* and $\overline{w}^*$ can be calculated using standard or modified Gauss procedure. The integral

$$\int_\Gamma \overline{\varphi}_n^* \cdot d\Gamma = \frac{1}{4\pi D} \cdot \int_\Gamma \left[\ln(r) + \frac{n^2}{r^2} \right] \cdot d\Gamma \tag{17}$$

where: $r = \sqrt{n^2 + s^2}$, may be evaluated approximately, assuming small curvature of element. Then, it is possible to assume that parameter n goes to zero. Hence, the function described by the equation (17) will have the form:

$$\overline{\varphi}_n^* \approx \frac{1}{4\pi D} \cdot \ln(r) \tag{18}$$

and integral of expression (18) may be evaluated analytically as integral on line element $\tilde{d}_i$ which is parallel to the s_i axe (Fig. 8).

For the edge resting on continuous linear support (simply-supported edge), integration of fundamental functions w^* and $\overline{w}^*$ can be done using classic or modified Gauss procedure (Fig. 6). The elements of characteristic matrix, which are responsive to integration of functions M_n^* and $\overline{M}_n^*$ may be calculated using method of rigid body movement. To calculate these elements, the boundary integral equation (1) must be considered. It is assumed, that all the non quasi-diagonal boundary integrals are calculated, external loading acting on a plate surface is equal to zero (p = 0, hence all of boundary variables are equal to zero) and the rigid rotation of plate is done (Fig. 7). Hence, the boundary integral equation (1) after discretization will have the form:

$$\sum_{k=1}^{le} \varphi_n^{(k)} \cdot \int_{\Gamma_k} M_n^* \cdot d\Gamma_k = 0 \quad \left(P^* = 1^*\right) \tag{19}$$

$$\sum_{k=1}^{le} \varphi_n^{(k)} \cdot \int_{\Gamma_k} \overline{M}_n^* \cdot d\Gamma_k = 0 \quad \left(M_n^* = 1^*\right) \tag{20}$$

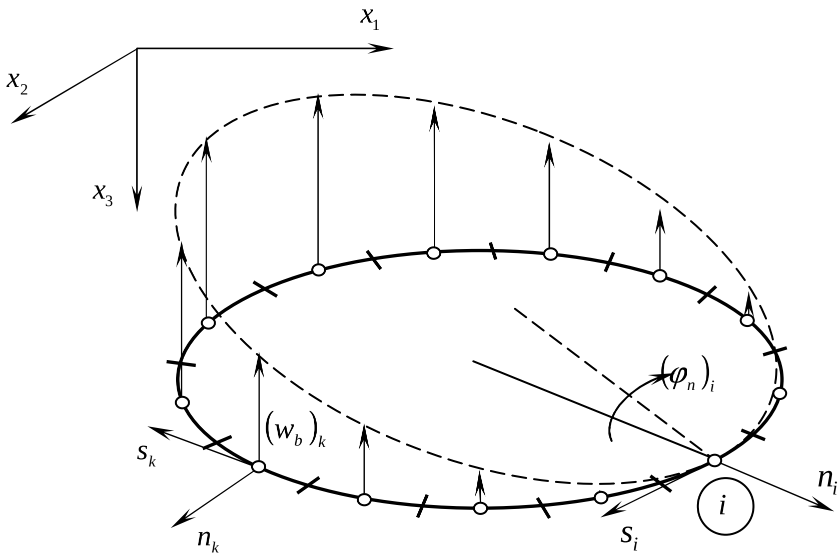

Fig. 7. Calculation of quasi-diagonal integrals using method of the rigid body movement: rigid rotation

For the free edge, integration of fundamental functions T_n^* and $\overline{T}_n^*$ can be calculated using rigid movement - rigid translation. It is assumed, that all the non quasi-diagonal boundary integrals are calculated, external loading acting on a plate surface is equal to zero ($p = 0$, hence all of boundary variables are equal to zero) and the rigid translation of plate is done: $w_b > 0$, $\varphi_n = 0$ and $\varphi_s = 0$. Hence, equation (1) after discretization will have the form:

$$\sum_{k=1}^{le} w_b^{(k)} \cdot \int_{\Gamma_k} T_n^* \cdot d\Gamma_k = 0 \quad \left(P^* = 1^*\right) \tag{21}$$

$$\sum_{k=1}^{le} w_b^{(k)} \cdot \int_{\Gamma_k} \overline{T}_n^* \cdot d\Gamma_k = 0 \quad \left(M_n^* = 1^*\right) \tag{22}$$

Angle of rotation in tangent direction φ_s on a plate boundary depends on boundary deflection w_b. Additional parameters φ_s, which are elements of matrix $\mathbf{\Delta}$, are calculated by construction difference expressions using deflection of three neighbouring physical nodes (Fig. 8). These expressions have the form:

$$\varphi_s^{(i-1)} = \frac{2d_{i+1} \cdot d_i \cdot w_b^{(i)} + d_{i+1}^2 \cdot w_b^{(i)} - d_i^2 \cdot w_b^{(i+1)} - 2d_i \cdot d_{i+1} \cdot w_b^{(i)} + d_i^2 \cdot w_b^{(i)} - d_{i+1}^2 \cdot w_b^{(i-1)}}{d_i \cdot d_{i+1} \cdot \left(d_i + d_{i+1}\right)} \tag{23}$$

$$\varphi_s^{(i)} = \frac{d_i^2 \cdot w_b^{(i+1)} - d_i^2 \cdot w_b^{(i)} + d_{i+1}^2 \cdot w_b^{(i)} - d_{i+1}^2 \cdot w_b^{(i-1)}}{d_i \cdot d_{i+1} \cdot \left(d_i + d_{i+1}\right)} \tag{24}$$

$$\varphi_s^{(i+1)} = \frac{d_i^2 \cdot w_b^{(i+1)} + 2d_i \cdot d_{i+1} \cdot w_b^{(i+1)} + d_{i+1}^2 \cdot w_b^{(i-1)} - d_i^2 \cdot w_b^{(i)} - 2d_{i+1} \cdot d_i \cdot w_b^{(i)} - d_{i+1}^2 \cdot w_b^{(i)}}{d_i \cdot d_{i+1} \cdot \left(d_i + d_{i+1}\right)} \tag{25}$$

In present formulation of boundary integral equations, the characteristic matrix **G** is fully populated. For large number of boundary elements and mixed boundary conditions, matrix **G** may be wrong conditioned.

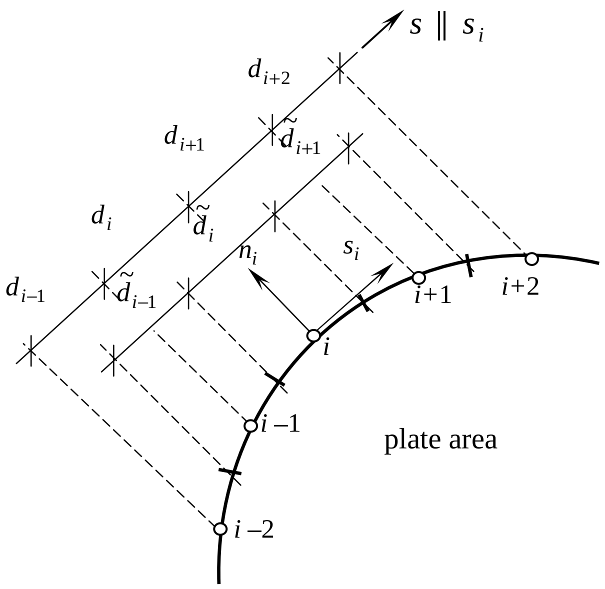

Fig. 8. Construction of difference expressions (23)-(25)

Expression (23) is applied for the first boundary element on a free edge. Expression (24) is used for all internal boundary elements: from second to last but one. Expression (25) is applied for the last boundary element taking place on a free edge.

4.2. Construction of right-hand-side vector

It is assumed, that constant loading p is acting on a plate surface. A contour of loading p was expressed in the poligonal form. Suitable integrals have the form: $p\int_{\Omega} w^* \cdot d\Omega$ and $p\int_{\Omega} \overline{w}^* \cdot d\Omega$. These integrals can be evaluated analytically according to Abdel-Akher and Hartley proposition [14].

5. Calculation of displacement and bending moments

Solution of the set of algebraic equation allowed to determine suitable boundary variables. Basing on the same boundary integral equation (1) it is possibile to calculate deflection at the arbitrary point of plate domain, hence coefficient $c(\mathbf{x})$ is equal to one. Plate deflection can be expressed as the sum of part depended on boundary variables and part depended on external loading:

$$w = w\left(\overline{\mathbf{B}}\right) + w\left(p\right) \tag{26}$$

The bending and twisting moments are defined according to thin plate theory - for example

$$M_{x_1}\left(x_1, x_2\right) = -D \cdot \left(\frac{\partial^2 w(x_1, x_2)}{\partial x_1^2} + \nu_{\mathrm{p}} \frac{\partial^2 w(x_1, x_2)}{\partial x_2^2}\right) \tag{27}$$

where $w(x_1, x_2)$ is the plate deflection at the point about coordinates x_1, x_2. Hence second derivatives of displacement take place in boundary integral equation (1). The bending moment may be expressed as the sum of parts connected with boundary variables and loading:

$$M_{x_i} = M_{x_i}\left(\overline{\mathbf{B}}\right) + M_{x_i}\left(p\right) \;\; i = 1, 2 \tag{28}$$

The twisting moment is expressed similar:

$$M_{x_1 x_2} = M_{x_1 x_2}\left(\overline{\mathbf{B}}\right) + M_{x_1 x_2}\left(p\right) \tag{29}$$

The parts of equations (28) and (29) connected with boundary variables can be evaluated numerically. Number of twelve Gauss point is applied in the analysis. Next, the integrals connected with external loading p acting on a plate surface must be considered: $\int\limits_{\Omega} \frac{\partial^2 w^*}{\partial^2 x_1} \cdot d\Omega$, $\int\limits_{\Omega} \frac{\partial^2 w^*}{\partial^2 x_2} \cdot d\Omega$ and $\int\limits_{\Omega} \frac{\partial^2 w^*}{\partial x_1 \partial x_2} \cdot d\Omega$. These integrals can be evaluated using method proposed by Abdel-Akher and Hartley [14].

6. Summary

Static analysis of thin plates by the boundary element method was presented in the paper. Physical boundary conditions were introduced. The boundary integral equations were formulated in singular and non-singular approach. In present formulation of plate bending it is no need to introduce equivalent shear forces at the plate edges and concentrated forces at the plate corners. It is element of originality in relation to classic formulation of thin plate bending problem. Presented approach may be also useful to static and dynamic of plates resting on internal supports:

pillars, linear and curvilinear continuous supports [3, 16, 17]. These types of bending problems can be solved using conception of Bèzine [3, 16], in which additional collocation points are applied inside the plate domain. The second part of elaboration includes numerical examples described static analysis of thin plates using curved, simplified boundary elements.

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