HELMHOLTZ EIGENVALUE PROBLEM IN ELLIPTICAL SHAPED DOMAINS

Stanisław Kukla

Institute of Mathematics, Czestochowa University of Technology, Poland
stanislaw.kukla@im.pcz.pl

Abstract. In this paper, the fundamental solutions method to the Helmholtz eigenvalue problem in two-dimensional elliptical shaped domains are presented. The Green's functions of the Helmholtz equation in the half-plane and in the quarter-plane are used. Numerical examples of the eigenvalue problems in a half-elliptic and a quarter-elliptic domains are given.

Introduction

The Helmholtz equation is obtained, for instance, by using separation method to the wave equation [1]. This equation can be written in the form

$$\nabla^2 f + \Omega^2 f = 0, \quad (x, y) \in S \tag{1}$$

where ∇^2 is the Laplace operator and S is the considered domain. In the case of initial-value problems which are governed by the unsteady diffusion equation as a result of separation of time and the space variables, the modified Helmholtz equation is acquired

$$\nabla^2 f - k^2 f = 0, \quad (x, y) \in S \tag{2}$$

The constants Ω and k in equations (1) and (2), respectively, are introduced by separation of variables. These equations are completed by conditions at the boundary ∂S of the domain S. We assume here the Dirichlet boundary condition

$$f(x, y) = 0, \quad (x, y) \in \partial S \tag{3}$$

The differential equation (1) or (2) and boundary condition (3) form the Helmholtz eigenvalue problems. The problem (1)-(3) for elliptic domain S (elliptical membrane) was the subject of the papers [1-3]. In this paper, we consider the half-elliptic and the quarter-elliptic domains. An approximate solution of the problems will be derived by using the fundamental solution method (MFS).

The MFS is a boundary method which does not involve discretization and integration. The idea of the method is the usage of a linear combination of fundamental solutions with sources located at fictitious points outside the domain of the problem. The fundamental solution is the Green's function G defined in an infinite domain. The functions $G(x, y; Q_k)$ satisfy the Helmholtz equation in the domain S for each source points $Q_k(\xi_k, \eta_k)$ located outside S. In the MFS, we approximate the solution of the problem by a function of the form [2]

$$w_n(x, y) = \sum_{k=1}^{n} c_k \, G(x, y; \xi_k, \eta_k) \tag{4}$$

The approximate solution w_n satisfies the differential equation (1), and it does not satisfy the boundary condition (3). The condition can be satisfied approximately by a suitable determination of the coefficients c_k, $k = 1, 2, \ldots, n$. For this purpose we use the least square method. First we choose the points $P_j(x_j, y_j)$, $j = 1, 2, \ldots, n$, located on boundary ∂S of the domain S. Next we definite the function

$$f((c_1, c_2, \ldots, c_n)) = \sum_{j=1}^{n} \left[\sum_{k=1}^{n} c_k \, G(P_j; Q_k) \right]^2 \tag{5}$$

This function has a minimum, if the following system of equations is satisfied

$$\mathbf{A}\,\mathbf{c} = \mathbf{0} \tag{6}$$

where $\mathbf{A} = [a_{ik}]_{1 \le i, k \le n}$, $a_{ik} = \sum_{j=1}^{n} G(P_j, Q_i) G(P_j, Q_k)$, $\mathbf{c} = [c_1 \quad c_2 \quad \ldots \quad c_n]^T$.

For a non-trivial solution of the system (6), the determinant of the matrix $\mathbf{A}$ is set equal to zero, yielding the eigenvalue equation

$$\det \mathbf{A}(\Omega) = 0 \tag{7}$$

Equation (7) with the unknown Ω, must be solved numerically to get the eigenvalues. The eigenfunctions for corresponding eigenvalues Ω_m, $m = 1, 2, \ldots$, are given by (4) where the coefficients c_k, $k = 2, \ldots, n$, are derived dependent on c_1 from n-1 equations of the system (6).

1. Fundamental solutions

The fundamental solution of the differential equation (1) in the half-plane: $-\infty < x < \infty$, $y \ge 0$, is a function (Green's function) G, which satisfies the following equation

$$\nabla^2 G + \Omega^2 G = \delta(x-\xi)\delta(y-\eta) \tag{8}$$

where δ(·) is the Dirac delta function. The solution of this equation with Dirichlet boundary condition: $f(x,0) = 0$, can be derived by using double Fourier transform. The transform is defined by the two relationships

$$\mathrm{F}[G] = \overline{\overline{G}}(\alpha,\beta,\xi,\eta) = \int_{-\infty}^{\infty}\int_{0}^{\infty} G(x,y,\xi,\eta) e^{i\alpha x} \sin\beta y \, dx\, dy \tag{9}$$

$$G(x,y,\xi,\eta) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} \overline{\overline{G}}(\alpha,\beta,\xi,\eta) e^{-i\alpha x} \sin\beta y \, d\alpha\, d\beta \tag{10}$$

where $\overline{\overline{G}}$ is the Fourier transform of the function *G*. If we multiply both sides of equation (1) by $e^{i\alpha x}\sin\beta y$, integrate over the half-plane: $-\infty < x < \infty,\ y \geq 0$ and use the properties of Fourier transform, we obtain the algebraic equation

$$\left(\alpha^2 + \beta^2 - \Omega^2\right)\overline{\overline{G}}(\alpha,\beta,\xi,\eta) = \frac{1}{2\pi} e^{i\alpha\xi} \sin\beta\eta \tag{11}$$

Using (11) in equation (10), we have

$$G(x,y,\xi,\eta) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{1}{\alpha^2 + \beta^2 - \Omega^2} e^{-i\alpha(x-\xi)} \sin\beta y \sin\beta\eta \, d\alpha\, d\beta \tag{12}$$

or after transformation

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\int_{0}^{\infty} \frac{1}{\sqrt{\beta^2 - \Omega^2}} e^{-(x-\xi)\sqrt{\beta^2-\Omega^2}} \left[\cos\beta(y-\eta) - \cos\beta(y+\eta)\right] d\beta \tag{13}$$

Finally, the Green's function for Helmholtz equation in the half-plane with Dirichlet boundary condition can be written in the form

$$G(x,y,\xi,\eta) = \frac{i}{4}\left[H_0^{(1)}\left(\Omega\sqrt{(x-\xi)^2 + (y-\eta)^2}\right) - H_0^{(1)}\left(\Omega\sqrt{(x-\xi)^2 + (y+\eta)^2}\right)\right] \tag{14}$$

where $H_0^{(1)}(\cdot)$ is the Hankel function of the first kind and zero order, $i = \sqrt{-1}$.

Similarly, the Green's function for Helmholtz equation (1) in the quarter-plane with boundary conditions: $G\big|_{x=0} = 0,\ G\big|_{y=0} = 0,$ can be obtained. The function has the form

$$G(x,y;\xi,\eta)=\frac{i}{4}\Big[H_0^{(1)}\Big(\Omega\sqrt{(x-\xi)^2+(y-\eta)^2}\Big)-H_0^{(1)}\Big(\Omega\sqrt{(x-\xi)^2+(y+\eta)^2}\Big)$$
$$-H_0^{(1)}\Big(\Omega\sqrt{(x+\xi)^2+(y-\eta)^2}\Big)+H_0^{(1)}\Big(\Omega\sqrt{(x+\xi)^2+(y+\eta)^2}\Big)\Big] \tag{15}$$

The Green's function (14) or (15) are used in eigenvalue equation (7).

2. Numerical examples

Applications of the fundamental solutions method with use free space Green's functions are widely presented in literature (for instance the papers [1-3]). In this paper, the method with using Green's functions which satisfy boundary conditions on a part of the edges of the considered domain is proposed. The function with free parameters as a solution of the differential equation is assumed. This function satisfies boundary conditions on a part of the edges. The presented numerical examples deal with eigenvalue problems for Helmholtz equation in half- or quarter-elliptic domains. The considered domains with source and collocation points are shown in Figure 1.

The Hankel function $H_0^{(1)}$, which occurs in equations (14)-(15), is the complex-valued function and that way the left hand side of the equation (7) takes the complex values. Therefore, we introduce a function F defined as

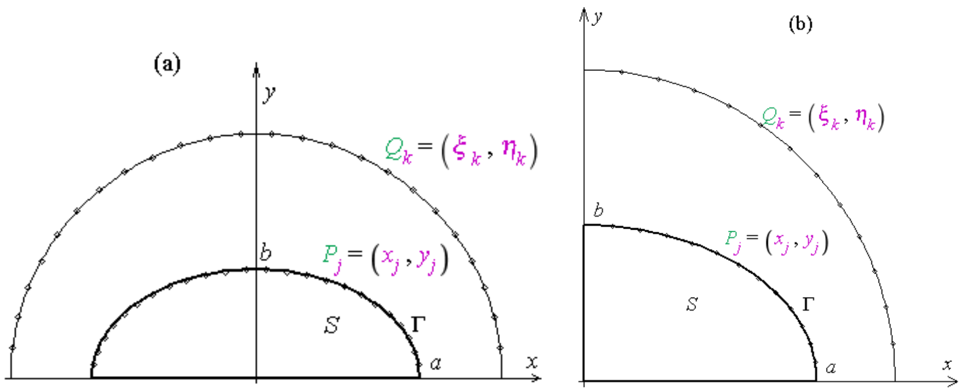

Fig. 1. Geometry configuration of the considered domains with collocation points P_j on the elliptical arch and source points Q_k on the circular arch; a) half-ellipse domain, b) quarter-ellipse domain

$$F(\Omega) = \left|\det \mathbf{A}(\Omega)\right| \tag{16}$$

where the symbol $|\cdot|$ denotes a modulus of a complex number. The minima of the function F determine roots of equation (7).

The eigenvalues of the Helmholtz operator are the frequency parameters of free vibration of a membrane. The first ten frequency parameter values Ω_n, n = 1, …10, of the half-elliptic membrane with clamped edges are presented in Table 1. The calculations were performed for various values of semi-diameters ratio a/b of the half-ellipse. For the half-circular membrane (a/b = 1.0) the frequency parameters determined by the FSM are compared with the exact eigenvalues which are obtained as roots of equation: $J_m(\Omega) = 0$, m = 1, 2,… . For assumed number of sources (n = 18), small differences of the results calculated by using MFS and exact values are observed.

Table 1

Eigenvalues Ω_n of the Helmholtz operator in a half-elliptic domain obtained by MFS for various values of semi-diameters ratio a/b

n	a/b = 1.0		a/b = 1.5	a/b = 2.0	a/b = 3.0
	FSM	Exact	FSM	FSM	FSM
1	3.83170	3.83171	3.54484	3.42588	3.32123
2	5.13562	5.13562	4.33781	3.99048	3.67965
3	6.38016	6.38016	5.16984	4.58509	4.05345
4	7.01559	7.01559	6.66668	6.55554	6.45773
5	7.58834	7.58834	6.02774	5.20393	4.44104
6	8.41724	8.41724	7.43162	7.09940	6.80660
7	8.77148	8.77148	6.90191	5.84208	4.84093
8	9.76102	9.76102	8.22539	7.66086	7.16374
9	9.93611	9.93610	7.78507	6.49552	5.25169
10	10.17347	10.17346	9.80190	9.69347	9.59771

In Table 2, the first ten frequency parameter values for one-quarter of the elliptic membrane with clamped edges are given. In FSM the Green's function for Helmholtz equation in the quarter-plane with Dirichlet boundary conditions was used. The results obtained for the circular sector by FSM are in agreement with exact ones.

Table 2

Eigenvalues Ω_n of the Helmholtz operator in a quarter-elliptic domain for various values of semi-diameters ratio *a*/*b*

n	a/b = 1.0 FSM	a/b = 1.0 Exact	a/b = 1.5 FSM	a/b = 2.0 FSM	a/b = 3.0 FSM
1	5.13562	5.13562	4.33781	3.99048	3.67965
2	7.58834	7.58834	6.02774	5.20390	4.44105
3	8.41724	8.41724	7.43162	7.09940	6.80660
4	9.93611	9.93611	7.78569	6.49544	5.25170
5	11.06471	11.06471	9.04446	8.23843	7.52872
6	11.61984	11.61984	10.55524	10.23049	9.94355
7	12.22510	12.22509	9.56317	7.83571	6.10097
8	13.58922	13.58929	10.74558	9.43641	8.28068
9	14.37240	14.37254	12.12377	11.34058	10.65190
10	14.79588	14.79595	13.68775	13.36692	13.08285

Conclusions

The Helmholtz eigenvalue problems in the half- and quarter-elliptic domains by using the method of fundamental solutions have been presented. The fundamental solution of the Helmholtz equation in the half-plane was derived. In order to determine the eigenvalues, the minimum of a real function was found. The source points occurring in the approximate formula of the solution were selected on a circle in the half-plane (or in the quarter-plane) outside the considered half-elliptic (quarter-elliptic) domain. The comparison of numerical results shows that high accuracy of the calculation is achieved for 18 sources.

References

[1] Kukla S., Method of fundamental solutions for Helmholtz eigenvalue problems in elliptical domains, Scientific Reserch of the Instiitute of Mathematics, 2009, 1(8), 85-90.

[2] Reutskiy S.Yu., The method of fundamental solutions for Helmholtz eigenvalue problems in simply and multiply connected domains, Engineering Analysis with Boundary Elements 2006, 30, 150-159.

[3] Civalek O., Gurses M., Free vibration of curvilinear membranes by eight-noded discrete singular convolution, International Journal of Science & Technology, 2008, 3(2), 165-171.

[4] Duffy D.G., Green's Functions with Applications, Chapman & Hall/CRC, Boca Raton, Washington DC 2001.