BOUNDARY ELEMENT METHOD FOR 3D PARABOLIC EQUATION - DETERMINATION OF FUNDAMENTAL SOLUTION

Joanna Klekot

Institute of Mathematics, Czestochowa University of Technology, Poland
joanna.drozdek@im.pcz.pl

Abstract. Parabolic equation with source term dependent on the first derivative of unknown function is considered. To solve this equation by means of the boundary element method the fundamental solution should be known. In the paper the fundamental solution for 3D problem is derived.

Introduction

The following parabolic equation is considered

$$\frac{\partial T(x,t)}{\partial t} = a\,\nabla^2 T(x,t) - \varepsilon\, u\,\frac{\partial T(x,t)}{\partial x_1} \tag{1}$$

where T is the temperature, $a = \lambda / c$ is the thermal diffusivity (λ is the thermal conductivity and c is the volumetric specific heat, respectively), u is the constant velocity, ε is the porosity, t denotes time, $x = \{x_1, x_2, x_3\}$ and

$$\nabla^2 T(x,t) = \sum_{e=1}^{3} \frac{\partial^2 T(x,t)}{\partial x_e^{\,2}} \tag{2}$$

The equation (1) is supplemented by boundary conditions

$$\begin{aligned} x \in \Gamma_1 &: \quad T(x,t) = T_b \\ x \in \Gamma_2 &: \quad q(x,t) = -\lambda\,\frac{\partial T(x,t)}{\partial n} = q_b \end{aligned} \tag{3}$$

and initial one

$$t = 0: \quad T(x,0) = T_p \tag{4}$$

where T_b and q_b are the known boundary temperature and boundary heat flux, respectively, T_p is the initial temperature, $\partial T / \partial n$ is the normal derivative

$$\frac{\partial T(x,t)}{\partial n} = \sum_{e=1}^{3} \frac{\partial T(x,t)}{\partial x_e} \cos\alpha_e \tag{5}$$

where $\cos\alpha_e$ are the directional cosines of the normal outward vector n.

The aim of investigations is to solve the problem formulated by means of the boundary element method. It is possible under the assumption that the fundamental solution is known. In this paper the fundamental solution is derived for 3D problem.

Fundamental solution

The equation (1) can be written in the form

$$\frac{1}{a}\frac{\partial T(x,t)}{\partial t} = \nabla^2 T(x,t) - \frac{\varepsilon u}{a}\frac{\partial T(x,t)}{\partial x_1} \tag{6}$$

At first, the following approximation with respect to time is proposed

$$\frac{1}{a}\frac{T(x,t^f)-T(x,t^{f-1})}{\Delta t} = \nabla^2 T(x,t^f) - \frac{\varepsilon u}{a}\frac{\partial T(x,t^f)}{\partial x_1} \tag{7}$$

this means

$$\nabla^2 T(x,t^f) - \frac{1}{a\,\Delta t} T(x,t^f) - \frac{\varepsilon u}{a}\frac{\partial T(x,t^f)}{\partial x_1} + \frac{1}{a\,\Delta t} T(x,t^{f-1}) = 0 \tag{8}$$

The weighted residual criterion for equation (8) has the following form [1, 2]

$$\int_\Omega \left[\nabla^2 T(x,t^f) - \frac{1}{a\,\Delta t} T(x,t^f) - \frac{\varepsilon u}{a}\frac{\partial T(x,t^f)}{\partial x_1} + \frac{1}{a\,\Delta t} T(x,t^{f-1})\right] T^*(\xi,x)\,\mathrm{d}\Omega = 0 \tag{9}$$

where $T^*(\xi,x)$ is the fundamental solution.

Using the second Green formula [1, 2] one has

$$\int_\Omega \nabla^2 T^*(\xi,x)\, T(x,t^f)\,\mathrm{d}\Omega + \int_\Gamma \left[T^*(\xi,x)\frac{\partial T(x,t^f)}{\partial n} - T(x,t^f)\frac{\partial T^*(\xi,x)}{\partial n}\right]\mathrm{d}\Gamma - \frac{1}{a\,\Delta t}\int_\Omega T(x,t^f)\, T^*(\xi,x)\,\mathrm{d}\Omega - \tag{10}$$

$$\int_\Omega \frac{\varepsilon u}{a}\frac{\partial T(x,t^f)}{\partial x_1}\, T^*(\xi,x)\,\mathrm{d}\Omega + \frac{1}{a\,\Delta t}\int_\Omega T(x,t^{f-1})\, T^*(\xi,x)\,\mathrm{d}\Omega = 0$$

or

$$\int_\Omega \left[\nabla^2 T^*(\xi,x) - \frac{1}{a\,\Delta t}\, T^*(\xi,x)\right] T(x,t^f)\,\mathrm{d}\Omega - \frac{1}{\lambda}\int_\Gamma T^*(\xi,x)\,q(x,t^f)\,\mathrm{d}\Gamma +$$
$$\frac{1}{\lambda}\int_\Gamma q^*(\xi,x)T(x,t^f)\,\mathrm{d}\Gamma + \int_\Omega \frac{\varepsilon u}{a}\frac{\partial T^*(x,t^f)}{\partial x_1}\, T(x,t^f)\,\mathrm{d}\Omega - \tag{11}$$
$$\int_\Gamma \frac{\varepsilon u}{a}\, T^*(\xi,x)T(x,t^f)\cos\alpha_1\,\mathrm{d}\Gamma + \frac{1}{a\,\Delta t}\int_\Omega T(x,t^{f-1})\, T^*(\xi,x)\mathrm{d}\Omega = 0$$

where $q(x,t^f) = -\lambda\,\partial T(x,t^f)/\partial n$ and $q^*(\xi,x) = -\lambda\,\partial T^*(\xi,x)/\partial n$.

Finally

$$\int_\Omega \left[\nabla^2 T^*(\xi,x) - \frac{1}{a\,\Delta t}\, T^*(\xi,x) + \frac{\varepsilon u}{a}\frac{\partial T^*(x,t^f)}{\partial x_1}\right] T(x,t^f)\,\mathrm{d}\Omega -$$
$$\frac{1}{\lambda}\int_\Gamma T^*(\xi,x)\,q(x,t^f)\,\mathrm{d}\Gamma + \frac{1}{\lambda}\int_\Gamma q^*(\xi,x)T(x,t^f)\,\mathrm{d}\Gamma - \tag{12}$$
$$\int_\Gamma \frac{\varepsilon u}{a}\, T^*(\xi,x)T(x,t^f)\cos\alpha_1\,\mathrm{d}\Gamma + \frac{1}{a\,\Delta t}\int_\Omega T(x,t^{f-1})\, T^*(\xi,x)\mathrm{d}\Omega = 0$$

It is visible that in the case considered the fundamental solution should fulfill the following equation

$$\nabla^2 T^*(\xi,x) - \frac{1}{a\,\Delta t}\, T^*(\xi,x) + \frac{\varepsilon u}{a}\frac{\partial T^*(x,t^f)}{\partial x_1} = -\delta(\xi,x) \tag{13}$$

where $\delta(\xi,x)$ is the Dirac function.

Because the fundamental solution for equation

$$\nabla^2 T^*(\xi,x) - \frac{1}{a\,\Delta t}\, T^*(\xi,x) = -\delta(\xi,x) \tag{14}$$

has the form [2]

$$T^*(\xi,x) = \frac{1}{4\pi r}\, e^{-\frac{r}{\sqrt{a\,\Delta t}}} \tag{15}$$

and the fundamental solution for equation

$$\nabla^2 T^*(\xi,x) + \frac{\varepsilon u}{a}\frac{\partial T^*(x,t^f)}{\partial x_1} = -\delta(\xi,x) \tag{16}$$

has the form [3]

$$T^*(\xi,x) = \frac{1}{4\pi\lambda r}\, e^{-\frac{\varepsilon u}{2a}\left[r+(x_1-\xi_1)\right]} \tag{17}$$

so for equation (13) the following fundamental solution is proposed

$$T^*(\xi,x) = \frac{1}{4\pi\lambda r}\, e^{-C_1\frac{1}{\sqrt{a\Delta t}}r \,-\, C_2\frac{\varepsilon u}{2a}\left[r+(x_1-\xi_1)\right]} \tag{18}$$

where C_1 and C_1 are the unknown constants.

The adequate derivatives are calculated

$$\frac{\partial T^*(\xi,x)}{\partial x_1} = \frac{1}{4\pi\lambda r}\, e^{-C_1\frac{1}{\sqrt{a\Delta t}}r \,-\, C_2\frac{\varepsilon u}{2a}\left[r+(x_1-\xi_1)\right]}\left[-\frac{x_1-\xi_1}{r^2} - \left(\frac{C_1}{\sqrt{a\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{x_1-\xi_1}{r} - \frac{C_2\,\varepsilon u}{2a}\right] \tag{19}$$

$$\frac{\partial T^*(\xi,x)}{\partial x_2} = \frac{1}{4\pi\lambda r}\, e^{-C_1\frac{1}{\sqrt{a\Delta t}}r \,-\, C_2\frac{\varepsilon u}{2a}\left[r+(x_1-\xi_1)\right]}\left[-\frac{x_2-\xi_2}{r^2} - \left(\frac{C_1}{\sqrt{a\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{x_2-\xi_2}{r}\right] \tag{20}$$

$$\frac{\partial T^*(\xi,x)}{\partial x_3} = \frac{1}{4\pi\lambda r}\, e^{-C_1\frac{1}{\sqrt{a\Delta t}}r \,-\, C_2\frac{\varepsilon u}{2a}\left[r+(x_1-\xi_1)\right]}\left[-\frac{x_3-\xi_3}{r^2} - \left(\frac{C_1}{\sqrt{a\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{x_3-\xi_3}{r}\right] \tag{21}$$

and

$$\frac{\partial^2 T^*(\xi,x)}{\partial x_1^{\,2}} = \frac{1}{4\pi\lambda r}\, e^{-C_1 \frac{1}{\sqrt{a\,\Delta t}} r \; - \; C_2 \frac{\varepsilon u}{2a}\left[r + (x_1-\xi_1)\right]} \left[3\, \frac{(x_1-\xi_1)^2}{r^4} + \right.$$

$$\left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)^2 \frac{(x_1-\xi_1)^2}{r^2} + 3\left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{(x_1-\xi_1)^2}{r^3} + \tag{22}$$

$$\left(\frac{C_2\,\varepsilon u}{2a}\right)^2 + \frac{C_2\,\varepsilon u}{a}\,\frac{x_1-\xi_1}{r^2} + \frac{C_2\,\varepsilon u}{a}\left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{x_1-\xi_1}{r} -$$

$$\left. \frac{1}{r^2} - \left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{1}{r} \right]$$

$$\frac{\partial^2 T^*(\xi,x)}{\partial x_2^{\,2}} = \frac{1}{4\pi\lambda r}\, e^{-C_1 \frac{1}{\sqrt{a\,\Delta t}} r \; - \; C_2 \frac{\varepsilon u}{2a}\left[r + (x_1-\xi_1)\right]} \left[3\, \frac{(x_2-\xi_2)^2}{r^4} + \right.$$

$$\left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)^2 \frac{(x_2-\xi_2)^2}{r^2} + 3\left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{(x_2-\xi_2)^2}{r^3} - \tag{23}$$

$$\left. \frac{1}{r^2} - \left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{1}{r} \right]$$

$$\frac{\partial^2 T^*(\xi,x)}{\partial x_3^{\,2}} = \frac{1}{4\pi\lambda r}\, e^{-C_1 \frac{1}{\sqrt{a\,\Delta t}} r \; - \; C_2 \frac{\varepsilon u}{2a}\left[r + (x_1-\xi_1)\right]} \left[3\, \frac{(x_3-\xi_3)^2}{r^4} + \right.$$

$$\left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)^2 \frac{(x_3-\xi_3)^2}{r^2} + 3\left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{(x_3-\xi_3)^2}{r^3} - \tag{24}$$

$$\left. \frac{1}{r^2} - \left(\frac{C_1}{\sqrt{a\,\Delta t}} + \frac{C_2\,\varepsilon u}{2a}\right)\frac{1}{r} \right]$$

Introducing (18), (19), (22), (23) and (24) into (13) one obtains

$$\begin{cases} C_1 = \dfrac{-\varepsilon u \Delta t + \sqrt{(\varepsilon u \Delta t)^2 + 4 a \Delta t}}{2\sqrt{a\Delta t}} \\ C_2 = 1 \end{cases} \tag{25}$$

So the searched fundamental solution has the form

$$T^*(\xi,x) = \frac{1}{4\pi\lambda r}\, e^{-\sqrt{\left(\frac{\varepsilon u}{2a}\right)^2 + \frac{1}{a\Delta t}}\, r \;-\; \frac{\varepsilon u}{2a}(x_1-\xi_1)} \tag{26}$$

The heat flux resulting from the fundamental solution is defined

$$q^*(\xi,x) = -\lambda \frac{\partial T^*(\xi,x)}{\partial n} \tag{27}$$

and this function can be calculated in analytical way

$$\begin{aligned} q^*(\xi,x) = \frac{\lambda}{4\pi r}\, e^{-\sqrt{\left(\frac{\varepsilon u}{2a}\right)^2 + \frac{1}{a\Delta t}}\, r \;-\; \frac{\varepsilon u}{2a}(x_1-\xi_1)} \Bigg\{ \Bigg[\sqrt{\left(\frac{\varepsilon u}{2a}\right)^2 + \frac{1}{a\Delta t}}\, \frac{x_1-\xi_1}{r} + \\ \frac{x_1-\xi_1}{r^2} + \frac{\varepsilon u}{2a} \Bigg] \cos\alpha_1 + \Bigg[\sqrt{\left(\frac{\varepsilon u}{2a}\right)^2 + \frac{1}{a\Delta t}}\, \frac{x_2-\xi_2}{r} + \frac{x_2-\xi_2}{r^2} \Bigg] \cos\alpha_2 \\ + \Bigg[\sqrt{\left(\frac{\varepsilon u}{2a}\right)^2 + \frac{1}{a\Delta t}}\, \frac{x_3-\xi_3}{r} + \frac{x_3-\xi_3}{r^2} \Bigg] \cos\alpha_3 \Bigg\} \end{aligned} \tag{28}$$

Summing up, the fundamental solution (26) allows one to apply the boundary element method for numerical solution of equation (1).

References

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