GOAL PROGRAMMING APPROACH FOR DERIVING PRIORITY VECTORS - SOME NEW IDEAS

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Abstract. The generation of priority vectors from pairwise comparison matrices is an essential part of the Analytic Hierarchy Process. Apart from the well-known Saaty's right eigenvalue method various other procedures have been proposed for priority modelling. Two most important alternative approaches are the statistical estimation techniques and methods based on constrained optimization models. In the paper a new goal programming model for deriving priority vectors and for measurement of consistency is proposed. In this approach the idea of goal programming is combined with the idea of Saaty's eigenvalue method. Some features of the method are studied via computer simulations.

Introduction

Analytic Hierarchy Process (AHP), since its invention, has been a tool at the hands of decision makers and researchers. Now it is one of the most widely used multiple criteria decision-making tools [1-3]. In that method, pairwise comparisons are performed by the decision-maker (DM) and then the pairwise comparison matrix (PCM) is a base for deriving the weights measuring the relative importance of the alternatives in the problem. The weights form so-called priority vector. By deriving priority vectors for all matrices in the hierarchy created for given decision problem, it is possible to perform standard AHP aggregation and obtain the final vector of the overall priorities. Thus generating priority vectors from pairwise comparison matrix is the core of the AHP. In developing the AHP Saaty [4, 5] suggested the right eigenvalue prioritization method (REM). In this approach, the priority vector is obtained as the normalized principal right eigenvector of the PCM corresponding to the largest eigenvalue which is simple and its existence is guaranteed by Perron's Theorem [6]. Although the REM remains perhaps the most popular prioritization technique, during last three decades a number of other methods have been proposed in the literature. Among them are the methods based on the statistical approach to the prioritization problem [7-9], and methods based on constrained optimization models [10-15]. However, each known method has its advantages and disadvantages and thus some literature argue that one method is better than another while other authors hold an opposite position. One can find in the literature number of papers devoted to comparative studies of various prioritization methods, [16-21]. Several authors suggest also combining different prioritization techniques to obtain better estimates of the underlying true priority vector, see e.g. [12, 22].

In the following section we state the prioritization problem formally. Section 2 briefly reviews the main prioritization approaches. Next we introduce a new method which combine the goal programming idea with the idea of the REM. This approach provide us not only with a new and relatively simple prioritization technique but also with intuitive measure of consistency of the decision-maker judgments. The last section presents the results of the simulation study in which our method is compared with the REM.

1. The problem - definitions and notation

A basic assumptions of the AHP is the existence of a unique (up to multiplying constant factor) natural vector of priorities $\mathbf{w} = (w_1,\ldots,w_n)'$ of n alternatives with respect to a given criterion. Commonly the priority weights w_i, $i = 1,\ldots,n$, are chosen to be nonnegative and normalized to unity: $\sum_i^n w_i = 1$. It is also assumed that priority ratios $a_{ij} = w_i/w_j$ can be translated into verbal expressions and that people are able to estimate these priority ratios with the help of translations rules. Tables with such rules proposed by Saaty are available in the AHP literature, see e.g. [5, 23]. Based on the assumptions, in the conventional AHP a decision-maker estimates ratios of priorities, which form the pairwise comparison matrix $\mathbf{A} = [a_{ij}]_{nxn}$. Typically, the input data of PCM is collected only for the upper triangle of the matrix $\mathbf{A}$, while the remaining elements are computed as the inverse of the corresponding symmetric elements in the upper triangle i.e. $a_{ij} = 1/a_{ji}$. Such a PCM is said to be *reciprocal* (RPCM). This method of data collection (leading to RPCM) forces some consistency of judgments which is not always natural, [7, 19, 21]. Some authors argue, see e.g. [7, 21], that enforcing the kind of consistency on the input data creates unnecessary dependency among observations and loses additional information contained perhaps in the elements of the lower triangle of $\mathbf{A}$ what and may lead to worse estimates of the priorities.

PCM is said to be *transitive* if the following condition holds: if an element a_{ij} is not less than an element a_{ik} then $a_{ij} \geq a_{ik}$ for $i = 1,\ldots,n$.

PCM is said to be *consistent* (CPCM) if it is reciprocal and its elements satisfy the condition: $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k = 1,\ldots,n$. A necessary and sufficient condition for a positive matrix $\mathbf{A}$ to be consistent is $a_{ij} = w_i/w_j$ $i, j = 1,\ldots,n$.

However, it is obvious that in reality it cannot be expected that the elements of PCM give exactly priority ratios. The human mind is not a perfect measurement device. Questions such as “compare - in a ratio scale - the importance of various features of your house”, or “tell me, how many times the student A is better in mathematics than the student B" do not have a precise answer. The answers (evaluations of the ratios) may depend on personal taste, experience, specific knowledge, the judge temporary mood and temper and may vary in time. We also cannot neglect rounding errors which can be quite big if we use integers and their inverses. Therefore typically, even if the comparisons are done very carefully, PCM is inconsistent and we have to express the relation between the PCM elements and the priority weights in the form

$$a_{ij} = e_{ij} \frac{w_i}{w_j} \tag{1}$$

where e_{ij} is a perturbation factor which is expected to be near 1. In the statistical approach the factor is interpreted as a realization of a random variable.

The fundamental problem of the AHP is how to determine the "true" priority vector given the PCM. Another problem connected with the theory is how to measure the degree of inconsistence of the PCM, because the significant violations of the consistency, may cause the usefulness of the data questionable.

2. Prioritization methods

We cannot expect to extract the true priority weights based on the given PCM, because of the perturbations in judgments, which can be considered as an uncontrollable and/or random process. Therefore we have to be content with good estimates. Several approaches for deriving the priorities will be listed and briefly described in this section.

2.1. Right eigenvalue method

The REM was fully developed, described and applied in the AHP context by Saaty [4, 5]. Now it is the first and most commonly used prioritization method.

In a perfect judgment case where are no perturbations ($e_{ij} = 1$) we have

$$\mathbf{Aw} = n\mathbf{w} \tag{2}$$

Thus in this case the priority vector **w** can be calculated by solving the eigenvector equation (2). It turns out that for a consistent matrix **A** the number n is the principal eigenvalue of **A**, i.e. the largest solution of the characteristic equation: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. It is also the only nonzero eigenvalue in this case. Saaty proposition when the matrix **A** is perturbed is to use the normalized right eigenvector associated with the largest eigenvalue as an estimate of the true priority vector. Thus to obtain the estimate we need to solve general eigenvector equation

$$\mathbf{Aw} = \lambda_{\max}\mathbf{w} \tag{3}$$

where $\lambda_{\max}$ is the principal eigenvalue. For an arbitrary positive reciprocal matrix **A** the value $\lambda_{\max}$ is always real, unique and not smaller than n.

Although the method has attracted much attention it also has been criticized in the literature. First of all it was design only for reciprocal matrices, and as we have mentioned earlier, the method of gathering information leading to RPCM is not natural and usually cause the loss of information. Secondly, for inconsistent matrices of order greater than 3 the solution is not invariant under transposition, see e.g. [16, 18]. This may cause difficulties in the interpretation of the weights values. Another criticism is connected with the fact that unlike most estimation procedures REM does optimize any criterion function and thus is difficult to interpret or to compare with other estimates, [11, 16]. Another share of criticism is connected with so called rank reversal phenomena, [23, 24]. It was also noticed that the Saaty's method is prone to influence of outliers in the data, [9, 25]. Finally, the REM requires complex calculations involving an iterative procedure, see e.g. [19].

2.2. Constrained optimization based methods (COBM)

COBM are generally the methods which look for a vector of weights **w** which produces a pairwise comparison matrix **M**(**w**), satisfies some conditions (such as positivity of coefficients, normalization etc.) and minimizes the distance to the given PCM, say **A**, with respect to a given criterion function measuring the distance. The most popular one is called the *logarithmic least squares method* (LLSM), [10-12, 19, 21]. In this method the criterion function measuring the distance between **A** and **M**(**w**) is given by

$$d(\mathbf{A},\mathbf{M}(\mathbf{w})) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\ln a_{ij} - \ln w_i + \ln w_j)^2$$

To obtain the estimate of the priority vector we minimize the above criterion subject to the conditions:

$$\prod_{i=1}^{n} w_i = 1, \quad w_i > 0, \quad i = 1,\ldots,n$$

Imposing the normalization condition $\sum_{i=1}^{n} w_i = 1$, the solution to this optimization problem has the following closed form:

$$w_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{1/n} \Bigg/ \sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}$$

Because of the form of the above solution the LLSM is also called a geometric means technique.

The second COBM is the *least square method* (LSM) [6, 19, 21]. The method can be presented as the following optimization problem

$$\min d(\mathbf{A},\mathbf{M}(\mathbf{w})) = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - w_i / w_j)^2$$

subject to

$$\sum_{i=1}^{n} w_i = 1, \quad w_i > 0, \quad i = 1,...,n$$

However, this method has its share of criticism and usually is not considered as an attractive alternative to REM or LLSM, [6, 19, 21].

Goal programming based methods form another group of methods belonging to COBM class. An interesting method implementing this approach was presented in [25]. This method sometimes is called *original goal programming method* (OGPM) and can be presented in the following form.

$$\min d(\mathbf{A},\mathbf{M}(\mathbf{v})) = \sum_{i=1}^{n}\sum_{j>i}^{n}(\ln\delta_{ij}^{+} + \ln\delta_{ij}^{-})$$

subject to

$$\begin{aligned}&\ln\delta_{ij}^{+} - \ln\delta_{ij}^{-} + \ln v_i - \ln v_j = \ln a_{ij}\\&\ln\delta_{ij}^{+}\cdot\ln\delta_{ij}^{-} = 0\\&\text{for all } 1 \le i < j \le n\end{aligned}$$

The solution to the problem is a vector $\mathbf{v} = (v_1,\dots,v_n)'$ which should be normalized to give the priority vector **w**. This method manifest some important features and is an interesting alternative to the most popular REM and LLSM, [12, 25]. Various modifications of this method and implementations of the goal programming idea in the AHP can be found in [13-15].

2.3. Other methods

In the statistical approach the elements of PCM are interpreted as a realizations of a random process. Commonly it is assumed that the random judgments have the form (2) and various prioritization methods obtained in this framework result from various assumptions about the distributions of the random perturbation factor, [7-9]. Usually the resulting estimators have not got a closed form solutions.

As a statistical method can also be obtained some of the above described COBM. For example LLSM can be derived as a maximum likelihood estimator when the random variables $Z_{ij} = \ln(e_{ij})$ are independent and normally distributed, see e.g. [16, 18]

Another approach for deriving priorities is based on fuzzy sets theory. In this approach the decision-maker express his/her opinions using fuzzy preference relation. A goal programming approach in this framework was considered in [26].

2.4. Measurement of consistency

There are several proposals in the literature for consistency measures. Among them are the Koczkodaj's inconsistency index [27], a residual mean square connected with LLSM [11, 16], and others, see e.g. [19, 25]. The natural measures of inconsistency in the case of optimization methods are simply the criterion functions *d* - however due to the variability of the optimization models and difficulties with intuitive interpretation of the criteria in that context they have not attracted much attention of the AHP community. One may observe that a popularity of a given consistency measure is closely related to the popularity of the prioritization method it is connected with. Thus the most popular measure is again connected with the REM and proposed by Saaty [5]. According to this concept the inconsistency of the data is measured as follows. First a consistency index CI(n) is computed as an average of all eigenvalues except the principal one. Thus, by applying well-known algebraic relations, the index CI(n) can be given in the following form [5, 19]:

$$\mathrm{CI}(n) = \frac{\lambda_{\max} - n}{n-1}$$

Next, the value of the index is compared with an average consistency index ACI(n) obtained from a sample of 500 of randomly generated reciprocal matrices of order n. The estimated values of ACI can be found in the AHP literature, see e.g. [5, 19]. Finally, Saaty proposed the so called consistency ratio CR = CI(n)/ ACI(n) for testing whether the information contained in the PCM is consistent enough to be is acceptable. Unfortunately, this index is constructed only for RPCM and even then it can be very misleading, see e.g. [23, 24].

3. Goal programming based priority vectors and consistency index

In this section we propose a new goal programming based prioritization method (GPPM) and connected with it a consistency index (GPCI) which can be used for generating the priority vectors from both reciprocal and nonreciprocal PCMs. The goal programming prioritization methods look for a vector of weights which gives a comparison matrix which minimizes the distance to the given PCM with respect to a given criterion function measuring the distance. In all known from the literature models the criterion functions compare directly the elements of the matrices and minimize an average of some measure of a distance between corresponding elements of the matrices **A** and **M(w)**. For example in the LLSM appropriate elements of the matrices are considered to be close to each other if a natural logarithm of their ratio is close to zero, or, equivalently, the ratio itself is close to one. The latter criterion is also employed in the OGPM while in the LSM we minimize the average of the squared Euclidian distances between corresponding elements of the matrices.

On the other hand REM look for a prioritization vector satisfying some kind of a generalization of the relation (2) holding in the perfect judgment case. We may say, that in this method the relation (2) is a base for the optimality criterion and thus we look for a vector "as close as possible" to the perfect one in a sense justified by the spectral theory. Here we propose to combine the concept that the optimal vector should approximately satisfy the relation (2) with the optimization approach. The idea is that if the human judgment are no perfect we should find a priority vector which satisfies the relation (2) as perfectly as possible. To achieve this aim we propose to estimate the priority vector by solving the following goal programming problem

$$\min \text{GPCI} = \frac{1}{n}\sum_{i=1}^{n}(d_i^+ + d_i^-)$$

subject to

$$d_i^+ - d_i^- + \sum_{j=1}^{n} a_{ij} w_j = n w_i$$

$$\sum_{j=1}^{n} w_j = 1, \quad w_i \ge 0, \quad d_i^+ \ge 0, \quad d_i^- \ge 0$$

$$i = 1,\dots,n$$

As a result we obtain the vector of priority weights as well as a new consistency index GPCI. In a consistent case GPCI always equals 0. For inconsistent PCMs the index takes a positive values. One can observe that in the perfect case the sum of the coefficients of the vectors $\mathbf{h_L} = \mathbf{Aw}$ and $\mathbf{h_R} = n\mathbf{w}$ (which are on left and right side of the equation (2) equals *n*. In inconsistent case the sum $\sum_{i=1}^{n}(d_i^+ + d_i^-)$ is simply the sum of absolute deviations between the coefficients of the two vectors: $\sum_{i=1}^{n}\left|(\mathbf{h_R} - \mathbf{h_L})_i\right|$. The index GPCI is equal to this sum divided by *n*. So, its value may be compared with 1, the sum of all priority weights, and, roughly speaking, it tells us what part of the mass of weights on the whole has to be changed (added or subtracted dependently on the coefficient) to achieve the equality. One can also easily verify, that in the case where the judgments of the decision-maker are consistent enough to indicate which of the alternatives is the best one, then the value of GPCI is not greater than 2–2/*n* (usually it is much smaller). Unlike the REM the proposed prioritization method and index can be applied to any type of PCM and thus provide us with a tool for dealing with nonreciprocal PCMs. In next section we compare the two approaches with the help of computer simulations.

4. Simulation studies results

In the simulations we generate the following types of random inconsistent PCMs:

– reciprocal (RPCM),
– reciprocal and transitive (RTPCM),
– transitive, nonreciprocal (TPCM).

To compare the methods REM and GPPM we generate 1000 matrices of each type. For each matrix we compute the priority vectors, $\mathbf{v}_{\mathrm{RE}}$ and $\mathbf{v}_{\mathrm{GP}}$ and consistency indices CR and GPCI, with the help of the REM and GPPM, respectively. Next we calculate:

– Pearson correlation coefficient between the priority vectors r_V,
– the rank correlation coefficient between the priority ranks ρ_V,
– the consistency indices CR and GPCI,
– the value of the goal programming criterion GPC for the Saaty's priority vector $\mathbf{v}_{\mathrm{RE}}$.

The latter is computed according the formula:

$$\mathrm{GPC}(\mathbf{v}) = \frac{1}{n}\sum_{i=1}^{n}\left|(\mathbf{A}\cdot\mathbf{v} - n\cdot\mathbf{v})_i\right|$$

Saaty in [5] propose an integers 1,…,9 and their inverses for evaluations of the preference ratios i.e. for the elements of PCMs. However this constraint is not a necessary requirement in AHP and many other authors propose much bigger variety of possible evaluation numbers, especially in the statistical approach, [7, 16, 17, 21]. Similarly as in [16], in our simulations the ratios are drown from the set of integers not greater than 50 and their inverses. The considered number of decision alternatives *n* is 5, 7 and 9.

Table 1 shows the mean correlation coefficients between priority vectors and between priority ranks generated with the help of two methods and based on the whole data set related to a given type of PCM. The table presents also the overall Pearson correlation coefficient between the consistency indices denoted by r_I.

Table 1

The results of comparative studies for REM and GPPM: the mean values of the correlation coefficients for different types of PCMs

	RPCM			RTPCM			TPCM		
	r_V	ρ_V	r_I	r_V	ρ_V	r_I	r_V	ρ_V	r_I
$n = 5$	0.999	0.990	0.642	0.998	1.00	0.568	0.990	0.970	0.707
$n = 7$	0.997	0.983	0.612	0.996	0.997	0.65	0.990	0.964	0.708
$n = 9$	0.995	0.975	0.590	0.996	0.995	0.704	0.988	0.963	0.650

Amazingly high correlation coefficients r_V and ρ_V confirm close relation between the two methods. The least values of the coefficients are obtained for the nonreciprocal PCMs and that results from the fact that the REM is designed for the reciprocal ones while the GPPM has much more general applications and can be adopted in all cases. The coefficient r_I has got smaller, yet still quite high values. It results from the fact, that GPPM often can find better explanation of the decision maker judgments and better justification for inconsistencies contained in the PCM. In other words the method can find vectors which better approximate the relation (2) than the REM. The study of the Table 2 enables deeper insight into the problem. It shows the mean values of the GPC for vectors obtained via both methods. The symbol RD stands for relative difference for the two values, i.e.

$$\text{RD} = [\text{GPC}(\mathbf{v}_{\text{RE}})\text{-GPC}(\mathbf{v}_{\text{GP}})]/\text{GPC}(\mathbf{v}_{\text{GP}})$$

Table 2

The results of comparative studies for REM and GPPM: the mean values of the GPC for different types of PCMs

	RPCM			RTPCM			TPCM		
	GPC_{RE}	GPC_{GP}	*RD*	GPC_{RE}	GPC_{GP}	*RD*	GPC_{RE}	GPC_{GP}	*RD*
$n = 5$	0.442	0.012	40.4	0.441	0.006	64.7	1.37	0.103	24.0
$n = 7$	0.681	0.022	34.6	0.503	0.007	67.1	1.16	0.059	38.6
$n = 9$	0.840	0.031	28.9	0.522	0.007	67.1	0.968	0.035	53.4

We see that GPPM provides us with priority vectors which much better approximate the relation (2).

There is also another interesting thing connected with new approach. The correlation coefficient between the Saaty's consistency index CR and the values of the goal programming criterion for the Saaty's priority vectors is equal to 1 in almost all considered data sets (containing 1000 records!). Only in the case of nonreciprocal PCMs the coefficient was between 0.995 and 0.9995 for n = 5,7,9.

Final remarks

The simulation studies show that the GPPM is an interesting alternative for the usual REM. It can be adopted for every type of PCM providing us with an consistency measure which enables us to compare all types of PCM with respect to this feature. The calculations necessary to generate the priority vector and the value of consistency index are rather easy - they can be performed with any standard calculation software such as commonly used spreadsheets. As a goal programming method it demonstrates also another good features. It provides the decision maker with a tool to comparison of various priority vectors generated by different me thods, i.e. the criterion function GPC(). Moreover it allows the decision maker to implement various other conditions which, according to decision-maker opinions, should be satisfied by the weights. For example this feature of the method can help us to prevent the rank reversal phenomena. We leave this investigation to future studies.

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