# ON A CERTAIN METHOD OF CALCULATING YOUNG MEASURES IN SOME SIMPLE CASES

***Piotr Puchała***

*Institute of Mathematics, Czestochowa University of Technology, Poland*
*p.st.puchala@gmail.com*

**Abstract.** We will explicitly calculate Young measure associated to a sequence that is uniformly bounded but not strongly convergent in $L^\infty(\Omega)$.

## 1. Motivation

One of the main problems concerned with direct methods of the calculus of variations is of the oscillation nature. There is a minimizing sequence to our problem and this sequence is uniformly bounded but is not strongly convergent. As it is bounded it has a subsequence converging weakly$*$ to a certain limit. As the indices grow the element of the subsequence oscillate more and more „wildly" around this weak* limit. However, passing to the limit erases much information about the oscillatory nature of the minimizing sequence because, roughly speaking, this weak* limit is a „mean value" of these oscillations. The idea of overcoming this disadvantage is to assign as a limit not a „usual" function but a probability measure - valued function called *Young measure* or *parametrized measure*. There are some difficulties in giving nontrivial examples of Young measures. Explicit calculations are based mainly on a generalized version of Riemann-Lebesgue lemma. The method presented in this paper, although needs some improvements, does not use this lemma, but only the change of variable theorem giving the same results as in examples in literature.

## 2. Definitions and notation

Let $\Omega$ be a nonempty, open bounded subset of $\mathbb{R}^n$. We denote by $L^\infty(\Omega)$ the space of essentially bounded functions on $\Omega$ with values in $\mathbb{R}^m$. We equip this space with the norm $\|\cdot\|_\infty$:

$$\|u\|_\infty := \text{esssup}\{|u(x)|: x \in \Omega\} = \inf\{0 \le b: |u(x)| \le b\}, \quad u \in L^\infty(\Omega)$$

where the inequality holds almost everywhere (a.e.) with respect to the Lebesgue measure.

Let $(u_n)$ be a sequence of functions belonging to $L^\infty(\Omega)$ such that:
(a) $\exists c > 0 \ \forall n \in \mathbb{N} \ \|u_n\|_\infty \le c < \infty$;
(b) $\forall n \in \mathbb{N} \ \forall x \in \Omega \ \ u_n(x) \in K \subset \mathbb{R}^m, K - \text{ compact}$;
(c) $u_n \xrightarrow[n\to\infty]{} u_\infty$ weakly* in $L^\infty(\Omega)$, that is $\forall w \in L^\infty(\Omega)$ we have

$$\int_\Omega w u_n \, dx \xrightarrow[n\to\infty]{} \int_\Omega w u_\infty \, dx$$

Let $h: \Omega \times K \to \mathbb{R}$ be a Carathéodory function (i.e. the function measurable with respect to the first variable and continuous with respect to the second variable). For any function $u \in L^\infty(\Omega)$ we set $(h \circ u)(x) := h(x, u(x))$. Sequence $(h \circ u_n)$ has a convergent subsequence (denoted by the same symbol for simplicity) convergent weakly* in $L^\infty(\Omega)$ to certain $g$, provided the above assumptions are fulfilled by sequence $(u_n)$ and function $h$.
In general $g \neq h \circ u_\infty$, but $g$ is a function with domain $\Omega$ and the range in the space of probability measures on $K \subset \mathbb{R}^m$. Thus,

$$g: \Omega \ni x \to g(x) \in {}^1(K).$$

Adopting the usual notation we will denote $g$ by $\nu$ and write $\nu_x \equiv \nu(x)$. So we have $\nu := (\nu_x)_{x\in\Omega}$. Measure $\nu$ is concentrated on set $\Omega \times K$.

Now, if $u_n \xrightarrow[n\to\infty]{} u_\infty$ weakly* in $L^\infty(\Omega)$, then $h \circ u_n \xrightarrow[n\to\infty]{} h \bullet \nu$ weakly* where we have

$$(h \bullet \nu)(x) := \int_K h(x,k) d\nu_x(k).$$

According to the Riesz representation theorem we can consider $\nu$ as a continuous linear functional on a space of Carathéodory integrands $\mathrm{Car}(\Omega, K)$

$$f: \mathrm{Car}(\Omega, K) \ni h \to f(h) := \int_\Omega (h \bullet \nu)(x) \, dx = \int_\Omega \int_K h(x,k) \, d\nu_x(k) \, dx \in \mathbb{R}.$$

We will call the parametrized probability measure $\nu = (\nu_x)_{x\in\Omega}$ a **Young measure**, while functional $f$ will be called **Young functional.**

We can associate a Young measure to any measurable function $u: \Omega \to \mathrm{K}$. This is the unique measure concentrated on the graph of $u$. It is an image of the Lebesgue measure $dx$ on $\Omega$ under the mapping $x \to (x, u(x))$.

## 3. Calculation of a Young measure

We consider the following integral:

$$\int_{\Omega\times K} h\,d\nu = \iint_{\Omega\ K} h(x,k)\,d\nu_x(k)\,dx.$$

Assume that $\Omega$ is nonempty, open and bounded subset of $\mathbb{R}$, $K$- nonempty compact subset of $\mathbb{R}$, $h:\Omega\times K\to\mathbb{R}$ - a Carathéodory function and let $u:\Omega\ni x\to u(x)\in K$ be a.e. differentiable function (with respect to the Lebesgue measure on $\Omega$). Further assume that function $h$ can be written as $h(x,k)=\alpha(x)\beta(k)$. Let us consider the inner integral on the right hand in the equation below:

$$\iint_{\Omega\ K} h(x,k)\,d\nu_x(k)\,dx = \int_{\Omega}\left[\int_K \beta(k)\,d\nu_x(k)\right]\alpha(x)dx$$

Substituting $k=u(x)$ we obtain

$$\int_K \beta(k)d\nu_x(k) = \int_{\Omega} \beta(u(x))\,dx$$

Applying the change of variable theorem once more we see that $\nu_x$ is a measure which is absolutely continuous with respect to the Lebesgue measure on $K$ and has density of the form $\frac{1}{u'(x)}$.

## 4. Examples

(1) Let $a,b\in\mathbb{R}_+,\Omega=(0,a)$ and consider the sequence of functions

$$u_n(x) := \begin{cases} \frac{2nb}{a}x-2bk, & x\in\left(\frac{ak}{n},\frac{(2k+1)a}{2n}\right] \\ -\frac{2nb}{a}x+2b(k+1), & x\in\left(\frac{(2k+1)a}{2n},\frac{(k+1)a}{n}\right) \end{cases}$$

where $n\in\mathbb{N}$ and $k=0,1,\dots,n-1$. Obviously, the range of each $u_n$ is the compact set $K=[0,b]$. Observe that for $n,m\in\mathbb{N}$ we have $\|u_n-u_m\|_\infty=b$, so sequence $(u_n)$ is not Cauchy in $L^\infty([0,a])$. This means that it does not converge strongly in $L^\infty((0,a))$. On the other hand it is uniformly bounded in $L^\infty((0,a))$, so it has a weakly* convergent in this space.

Let us fix $n \in \mathbb{N}$ and apply our procedure to sequence $(u_n)$. Using the change of variable theorem to the inner integral we get

$$\int_0^b \beta(k)\, dv_x(k) = \int_0^a \beta(u_n(x))\, dx =$$

$$= \sum_{k=0}^{n-1} \int_{\frac{ak}{n}}^{\frac{(2k+1)a}{2n}} \beta\left(\tfrac{2nb}{a}x - 2bk\right) dx + \sum_{k=0}^{n-1} \int_{\frac{(2k+1)a}{2n}}^{\frac{(k+1)a}{n}} \beta\left(-\tfrac{2nb}{a}x + 2b(k+1)\right) dx =$$

$$= I_1 + I_2.$$

We consider the k-th integral in $I_1$. Substitution $\frac{2nb}{a}x - 2bk = z$ yields the formula

$$\int_{\frac{ak}{n}}^{\frac{(2k+1)a}{2n}} \beta\left(\tfrac{2nb}{a}x - 2bk\right) dx = \frac{1}{2n}\int_0^b \beta(z)\frac{a}{b}\, dz$$

There are $n$ integrals in $I_1$, so $I_1 = \frac{1}{2}\int_0^b \beta(z)\frac{a}{b}dz$.

Analogously, $I_2 = \frac{1}{2}\int_0^b \beta(z)\frac{a}{b}dz$. Finally, we obtain

$$\int_0^b \beta(k) dv_x(k) = \int_0^b \beta(z)\frac{a}{b}\, dz$$

Thus, we conclude that the Young measure associated to sequence $u_n$ has the form

$$v_x = \frac{a}{b}dz$$

i.e. it is a homogeneous Young measure (that is independent of $x \in \Omega$) and is absolutely continuous with respect to the Lebesgue measure on $K$ with the density equal to $\frac{a}{b}$.

(2) Let $\Omega = (0,1)$ and consider function $u_1 : \Omega \to \mathbb{R}$ (see [3] p.115):

$$u_1(x) := \begin{cases} 3x, & x \in \left(0, \frac{1}{6}\right] \\ \frac{3}{2}x + \frac{1}{4}, & x \in \left(\frac{1}{6}, \frac{1}{2}\right] \\ -\frac{3}{2}x + \frac{7}{4}, & x \in \left(\frac{1}{2}, \frac{5}{6}\right] \\ -3x + 3, & x \in \left(\frac{5}{6}, 1\right) \end{cases}.$$

We see that $u_1(\Omega) = [0,1]$. Let $(u_n)$ be a sequence of functions with the first element $u_1$: one "tooth" (i.e. graph of $u_1$) in $\Omega$; $u_n - n$ such shaped "teeth" in $(0,1)$. Applying procedure described above to the defined functions we see that Young measure $\nu_x$ associated to the function $u_n$ is absolutely continuous with respect to the Lebesgue measure on $K$ with the density $f_{\nu_x}$ equal to

$$f_{\nu_x}(k) = \begin{cases} \frac{2}{3}, k \in \left[0, \frac{1}{2}\right] \\ \frac{4}{3}, k \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

**Remark I**

Let us observe that in the above examples we fixed $n$ arbitrarily, so the sequence of Young measures associated to our sequence of functions is constant and as such it is trivially convergent to the calculated Young measure. It allows us to ask whether in some special cases the sequences of Young measures associated to the sequences of measurable functions are the constant ones. It is also of interest to characterize the class of all those sequences of measurable functions whose sequences of associated Young measures are the constant ones or such that their limits can be calculated easily.

**Remark II**

The Young measures appears in a natural way in engineering problems. In the calculus of variations or in the static problems in mechanics the basic role is played by the integral functionals of the type

$$J(u) = \int_\Omega f\big(x, u(x), \nabla u(d)\big)\, dx$$

acting on an appropriate function space. One of the most effective methods of seeking minima of such functionals is the direct method. It allows to investigate the minimum of the functional without analyzing the Euler - Lagrange equations. However, even the convexity of the integrand $f$ (or even some generalizations of the notion of convexity) does not guarantee the existence of the classical minimizers for $J$. In this case minimizing sequences oscillate violently around their weak limits. The procedure of convexyfying the integrand, although widely used, erases much information about the oscillatory nature of the minimizing sequences. The Young measures can be effectively used to analyze the oscillation phenomena.

Let us mention that in engineering we often meet nonconvex integrands; in smart materials the density of the internal energy is not a convex function (in fact, it is not even quasiconvex). The lack of convexity appears also in crystal twinning: in crystals the density of the internal energy is a many - well potential, obviously a nonconvex function.

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## References

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[2] Pedregal P., Variational Methods in Nonlinear Elasticity, Society for Industrial and Applied Mathematics, Philadelphia 2000.

[3] Roubíček T., Relaxation in Optimization Theory and Variational Calculus Walter de Gruyter, Berlin, New York 1997.

## ERRATUM

# Errata to „On the queue-length distribution in the $GI^X/G/1$ system with server vacations and exhaustive service"

[Scientific Research of the Institute of Mathematics and Computer Science 2(7), 2008, p. 23-30]

Wojciech M. Kempa

*Silesian University of Technology, Gliwice, Poland*

The analysis is invalid for general-type distributions of interarrival times. It should be restricted to models with Poisson arrivals (thus to systems of $M^X/G/1$ type) due to significance of memoryless property. All other distributions (of service times, bath sizes and vacations) can be of general types.