ADDITIVITY OF THE TANGENCY RELATION OF RECTIFIABLE ARCS

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Abstract. In this paper the problem of the additivity of some tangency relation of sets for the rectifiable arcs in the generalized metric spaces is considered. Some sufficient conditions for the additivity of this relation are given.

Introduction

Let E be an arbitrary non-empty set, and E_0 the family of all non-empty subsets of set E. Let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$, and let l_0 be the function of the form:

$$l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for} \quad x, y \in E \tag{1}$$

We shall call pair (E, l) the generalized metric space (see [1]). By some assumptions relating to the function l, function l_0 defined by formula (1) will be the metric of set E.

Using (1) we may define in the space (E, l), similarly as in a metric space, the following notions: sphere $S_l(p, r)$ and open ball $K_l(q, u)$

$$S_l(p, r) = \{x \in E : l_0(p, x) = r\} \text{ and } K_l(q, u) = \{x \in E : l_0(q, x) < u\} \tag{2}$$

Let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0 \tag{3}$$

We will denote by $S_l(p, r)_u$ (see [2, 3]) the u-neighbourhood of sphere $S_l(p, r)$ in space (E, l) defined by the following formula:

$$S_l(p, r)_u = \begin{cases} \bigcup\limits_{q \in S_l(p,r)} K_l(q, u) & \text{for} \quad u > 0 \\ S_l(p, r) & \text{for} \quad u = 0 \end{cases} \tag{4}$$

We say that pair (A, B) of sets $A, B \in E_0$ is (a, b)-clustered at point p of space (E, l), if 0 is the cluster point of the set of all real numbers $r > 0$ such that $A \cap S_l(p, r)_{a(r)} \neq \emptyset$ and $B \cap S_l(p, r)_{a(r)} \neq \emptyset$.

Let us define the following set (see [1, 3, 4])

$$T_l(a, b, k, p) = \{(A, B) : \ A, B \in E_0, \text{ pair } (A, B) \text{ is } (a, b)\text{-clustered at point } p \text{ of space } (E, l) \text{ and } \frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0\} \tag{5}$$

If pair $(A, B) \in T_l(a, b, k, p)$, then we say that set $A \in E_0$ is (a, b)-tangent of order $k > 0$ to set $B \in E_0$ at point p of the generalizd metric space (E, l).

Set $T_l(a, b, k, p)$ defined by formula (5) is called the (a, b)-tangency relation of order k of sets at point p in the generalized metric space (E, l).

Let ρ be a metric of set E and let A, B be arbitrary sets of the family E_0. Let us denote

$$\rho(A, B) = \inf\{\rho(x, y) : x \in A, \ y \in B\} \tag{6}$$

$$d_\rho A = \sup\{\rho(x, y) : x, y \in A\} \tag{7}$$

We shall denote by $\mathfrak{F}_\rho$ the class of all functions l fulfilling the conditions:

1^0 $\ l : E_0 \times E_0 \longrightarrow [0, \infty)$,

2^0 $\ \rho(A, B) \leq l(A, B) \leq d_\rho(A \cup B) \quad \text{for} \quad A, B \in E_0$.

From equality (1) and from condition 2^0 it follows that

$$l(\{x\}, \{y\}) = l_0(x, y) = \rho(x, y) \ \text{ for } \ l \in \mathfrak{F}_\rho \ \text{ and } \ x, y \in E \tag{8}$$

The above equality implies that any function $l \in \mathfrak{F}_\rho$ generates in set E the metric ρ.

Let $\widetilde{A}_p$ be the class of the rectifiable arcs with the origin at point $p \in E$ of the form (see [5, 6, 7]):

$$\widetilde{A}_p = \{A \in E_0\colon \lim_{A \ni x \to p} \frac{\ell(\widetilde{px})}{\rho(p, x)} = g < \infty\} \tag{9}$$

where $\ell(\widetilde{px})$ denotes the length of the arc $\widetilde{px}$ with ends p and x.

Let l_1, l_2 be arbitrary functions belonging to the class $\mathfrak{F}_\rho$. For these functions we define their sum as follows

$$(l_1 + l_2)(A, B) = l_1(A, B) + l_2(A, B) \quad \text{for} \quad A, B \in E_0 \tag{10}$$

We say that tangency relation $T_l(a, b, k, p)$ is additive in the class of functions $\mathfrak{F}_\rho$ if

$(A, B) \in T_{l_1+l_2}(a, b, k, p)$ if and only if $(A, B) \in T_{l_1}(a, b, k, p) \cup T_{l_2}(a, b, k, p)$ for $l_1, l_2 \in \mathfrak{F}_\rho$ and $A, B \in E_0$.

In this paper the problem of the additivity of the tangency relation $T_l(a, b, k, p)$ of the rectifiable arcs belonging to the class $\widetilde{A}_p$ in the spaces (E, l), for the functions $l \in \mathfrak{F}_\rho$ is considered. Some sufficient conditions for the additivity of this tangency relation shall be given.

1. On the additivity of the tangency relation of the arcs

Let l_1, l_2 be the functions belonging to class $\mathfrak{F}_\rho$ and E be any non-empty set.

Lemma 1.1. *If functions $l_1, l_2 \in \mathfrak{F}_\rho$, then*

$$S_{l_1+l_2}(p, r)_u = S_\rho(p, r/2)_{u/2} \tag{11}$$

`Proof.` From equalities (8) and (10) we have

$$K_{l_1+l_2}(p, r) = \{x \in E : \ (l_1 + l_2)(\{p\}, \{x\}) < r\}$$
$$= \{x \in E : \ l_1(\{p\}, \{x\}) + l_2(\{p\}, \{x\}) < r\} = \{x \in E : \ 2\rho(p, x) < r\}$$
$$= \{x \in E : \ \rho(p, x) < r/2\} = K_\rho(p, r/2).$$

Therefore

$$K_{l_1+l_2}(p, r) = K_\rho(p, r/2) \quad \text{for} \quad l_1, l_2 \in \mathfrak{F}_\rho \tag{12}$$

Similarly

$$S_{l_1+l_2}(p, r) = S_\rho(p, r/2) \quad \text{for} \quad l_1, l_2 \in \mathfrak{F}_\rho \tag{13}$$

From definition (4) of the u-neighbourhood $S_l(p, r)_u$ of sphere $S_l(p, r)$, and from formulas (12) and (13) we obtain thesis (11) of the above lemma.

An immediate consequence of (13) is the following equality

$$S_{l_1+\cdots+l_n}(p, r)_u = S_\rho(p, r/n)_{u/n} \quad \text{for} \quad l_1, \ldots, l_n \in \mathfrak{F}_\rho \tag{14}$$

From Lemma 2.2 of the paper [2] (see also [8]) we get the following corollary:

Corollary 1.1. *If function a fulfils the condition*

$$\frac{a(r)}{r} \xrightarrow[r\to 0^+]{} 0 \tag{15}$$

then for an arbitrary arc $A \in \widetilde{A}_p$

$$\frac{1}{r} d_\rho(A \cap S_\rho(p, r/n)_{a(r)/n}) \xrightarrow[r\to 0^+]{} 0 \tag{16}$$

From condition (16) it follows immediately that

$$\frac{1}{r} d_\rho(A \cap S_\rho(p, r/2)_{a(r)/2}) \xrightarrow[r\to 0^+]{} 0 \quad \text{for } A \in \widetilde{A}_p \tag{17}$$

Now using these considerations we prove:

Theorem 1.1. *If the non-decreasing functions* a, b *fulfil the condition*

$$\frac{a(r)}{r} \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r} \xrightarrow[r\to 0^+]{} 0 \tag{18}$$

and $l_1, l_2 \in \mathfrak{F}_\rho$, *then*

$(A, B) \in T_{l_1}(a, b, 1, p) \cup T_{l_2}(a, b, 1, p)$ *if and only if* $(A, B) \in T_{l_1+l_2}(a, b, 1, p)$

for arcs $A, B \in \widetilde{A}_p$.

`Proof.`We assume that $(A, B) \in T_{l_1}(a, b, 1, p) \cup T_{l_2}(a, b, 1, p)$. Hence in particular results that $(A, B) \in T_{l_1}(a, b, 1, p)$ for $A, B \in \widetilde{A}_p$. From this we obtain

$$\frac{1}{r} l_1(A \cap S_\rho(p, r/2)_{a(r/2)}, B \cap S_\rho(p, r/2)_{b(r/2)}) \xrightarrow[r\to 0^+]{} 0 \tag{19}$$

Hence and from the fact that $l_1 \in \mathfrak{F}_\rho$ we get

$$\frac{1}{r} \rho(A \cap S_\rho(p, r/2)_{a(r/2)}, B \cap S_\rho(p, r/2)_{b(r/2)}) \xrightarrow[r\to 0^+]{} 0 \tag{20}$$

The properties of functions a, b imply the following inequality

$$\begin{aligned} 0 &\le \rho(A \cap S_\rho(p, r/2)_{a(r)}, B \cap S_\rho(p, r/2)_{b(r)}) \\ &\le \rho(A \cap S_\rho(p, r/2)_{a(r/2)}, B \cap S_\rho(p, r/2)_{b(r/2)}) \end{aligned}$$

for arbitrary sets $A, B \in E_0$.

Thus, we can apply condition (20) and obtain

$$\frac{1}{r}\rho(A\cap S_\rho(p,r/2)_{a(r)},B\cap S_\rho(p,r/2)_{b(r)})\xrightarrow[r\to 0^+]{}0 \tag{21}$$

Hence and from theorem on the compatibility of the tangency relations of sets of the class $A^*_{p,1}\supset \widetilde{A}_p$ (see Theorem 3 in paper [4]) we get

$$\frac{1}{r}d_\rho((A\cap S_\rho(p,r/2)_{a(r)})\cup(B\cap S_\rho(p,r/2)_{b(r)}))\xrightarrow[r\to 0^+]{}0 \tag{22}$$

As the following inequality

$$\begin{aligned}0&\le d_\rho((A\cap S_\rho(p,r/2)_{a(r)/2})\cup(B\cap S_\rho(p,r/2)_{b(r)/2}))\\&\le d_\rho((A\cap S_\rho(p,r/2)_{a(r)})\cup(B\cap S_\rho(p,r/2)_{b(r)})),\end{aligned}$$

is valid for any two sets $A,B\in E_0$, then from formula (22) it follows

$$\frac{1}{r}d_\rho((A\cap S_\rho(p,r/2)_{a(r)/2})\cup(B\cap S_\rho(p,r/2)_{b(r)/2}))\xrightarrow[r\to 0^+]{}0.$$

Hence we get

$$\frac{1}{r}l_1(A\cap S_\rho(p,r/2)_{a(r)/2},B\cap S_\rho(p,r/2)_{b(r)/2})\xrightarrow[r\to 0^+]{}0 \tag{23}$$

for an arbitrary function $l_1\in\mathfrak{F}_\rho$.

From the fact that $l_1,l_2\in\mathfrak{F}_\rho$ and from Lemma 1.1 we obtain

$$\begin{aligned}&(l_1+l_2)(A\cap S_{l_1+l_2}(p,r)_{a(r)},B\cap S_{l_1+l_2}(p,r)_{b(r)})\\&=l_1(A\cap S_{l_1+l_2}(p,r)_{a(r)},B\cap S_{l_1+l_2}(p,r)_{b(r)})\\&+l_2(A\cap S_{l_1+l_2}(p,r)_{a(r)},B\cap S_{l_1+l_2}(p,r)_{b(r)})\\&=l_1(A\cap S_\rho(p,r/2)_{a(r)/2},B\cap S_\rho(p,r/2)_{b(r)/2})\\&+l_2(A\cap S_\rho(p,r/2)_{a(r)/2},B\cap S_\rho(p,r/2)_{b(r)/2})\\&\le 2d_\rho((A\cap S_\rho(p,r/2)_{a(r)/2})\cup(B\cap S_\rho(p,r/2)_{b(r)/2}))\\&\le 2d_\rho((A\cap S_\rho(p,r/2)_{a(r)/2})+2d_\rho((B\cap S_l(p,r/2)_{b(r)/2})\\&+2\rho(A\cap S_\rho(p,r/2)_{a(r)/2},B\cap S_\rho(p,r/2)_{b(r)/2})\\&\le 2d_\rho((A\cap S_\rho(p,r/2)_{a(r)/2})+2d_\rho((B\cap S_\rho(p,r/2)_{b(r)/2})\\&+2l_1(A\cap S_\rho(p,r/2)_{a(r)/2},B\cap S_\rho(p,r/2)_{b(r)/2}),\end{aligned}$$

whence

$$\begin{aligned}
&\frac{1}{r}(l_1+l_2)(A\cap S_{l_1+l_2}(p,r)_{a(r)}, B\cap S_{l_1+l_2}(p,r)_{b(r)})\\
\le\ &\frac{2}{r}d_\rho((A\cap S_\rho(p,r/2)_{a(r)/2})+\frac{2}{r}d_\rho((B\cap S_\rho(p,r/2)_{b(r)/2})\\
&+\frac{2}{r}l_1(A\cap S_\rho(p,r/2)_{a(r)/2}, B\cap S_\rho(p,r/2)_{b(r)/2})
\end{aligned}\tag{24}$$

Formulas (23), (24) and Corollary 1.1 imply the relation

$$\frac{1}{r}(l_1+l_2)(A\cap S_{l_1+l_2}(p,r)_{a(r)}, B\cap S_{l_1+l_2}(p,r)_{b(r)})\xrightarrow[r\to 0^+]{}0 \tag{25}$$

valid for $A,B\in\widetilde{A}_p$ and $l_1,l_2\in\mathfrak{F}_\rho$.
From the fact that $A,B\in\widetilde{A}_p$ it follows that pair of sets (A,B) is (a,b)-clustered at point p of space (E,l_1+l_2) for $l_1,l_2\in\mathfrak{F}_\rho$. We conclude that $(A,B)\in T_{l_1+l_2}(a,b,1,p)$.

Now we assume that $(A,B)\in T_{l_1+l_2}(a,b,1,p)$ for $A,B\in\widetilde{A}_p$. Hence we have

$$\frac{1}{r}(l_1+l_2)(A\cap S_{l_1+l_2}(p,r)_{a(r)}, B\cap S_{l_1+l_2}(p,r)_{b(r)})\xrightarrow[r\to 0^+]{}0.$$

From the above relation and from definition (10) it follows that

$$\frac{1}{r}l_1(A\cap S_{l_1+l_2}(p,r)_{a(r)}, B\cap S_{l_1+l_2}(p,r)_{b(r)})\xrightarrow[r\to 0^+]{}0 \tag{26}$$

Thus, we apply Lemma 1.1 and (26) we get

$$\frac{1}{r}l_1(A\cap S_\rho(p,r/2)_{a(r)/2}, B\cap S_\rho(p,r/2)_{b(r)/2})\xrightarrow[r\to 0^+]{}0.$$

The assumption: $l_1\in\mathfrak{F}_\rho$ yields the following formula

$$\frac{1}{r}\rho(A\cap S_\rho(p,r/2)_{a(r)/2}, B\cap S_\rho(p,r/2)_{b(r)/2})\xrightarrow[r\to 0^+]{}0 \tag{27}$$

As inequality

$$\begin{aligned}
0\ &\le\ \rho(A\cap S_\rho(p,r/2)_{a(r)}, B\cap S_\rho(p,r/2)_{b(r)})\\
&\le\ \rho(A\cap S_\rho(p,r/2)_{a(r)/2}, B\cap S_\rho(p,r/2)_{b(r)/2})
\end{aligned}$$

is valid for arbitrary sets $A,B\in E_0$, then from formula (27) it follows that

$$\frac{1}{r}\rho(A\cap S_\rho(p,r/2)_{a(r)}, B\cap S_\rho(p,r/2)_{b(r)})\xrightarrow[r\to 0^+]{}0 \tag{28}$$

Hence we get

$$\frac{1}{r} d_\rho((A \cap S_\rho(p, r/2)_{a(r)}) \cup (B \cap S_\rho(p, r/2)_{b(r)})) \xrightarrow[r \to 0^+]{} 0 \tag{29}$$

As the following inequality

$$\begin{aligned} 0 &\leq d_\rho((A \cap S_\rho(p, r/2)_{a(r/2)}) \cup (B \cap S_\rho(p, r/2)_{b(r/2)})) \\ &\leq d_\rho((A \cap S_l(p, r/2)_{a(r)}) \cup (B \cap S_l(p, r/2)_{b(r)})) \end{aligned}$$

is fulfilled for $A, B \in E_0$, then from frmula (29) it follows

$$\frac{1}{r^k} d_\rho((A \cap S_\rho(p, r/2)_{a(r/2)}) \cup (B \cap S_\rho(p, r/2)_{b(r/2)})) \xrightarrow[r \to 0^+]{} 0 \tag{30}$$

Applying assumption: $l_1 \in \mathfrak{F}_\rho$ we obtain

$$\frac{1}{r} l_1(A \cap S_\rho(p, r/2)_{a(r/2)}, B \cap S_\rho(p, r/2)_{b(r/2)}) \xrightarrow[r \to 0^+]{} 0,$$

i.e.

$$\frac{1}{t} l_1(A \cap S_\rho(p, t)_{a(t)}, B \cap S_\rho(p, t)_{b(t)}) \xrightarrow[t \to 0^+]{} 0 \tag{31}$$

From the fact that pair of arcs (A, B) is (a, b)-clustered at point p of space (E, l_1) and from condition (31) it follows that $(A, B) \in T_{l_1}(a, b, 1, p)$.Therefore $(A, B) \in T_{l_1}(a, b, k, p) \cup T_{l_2}(a, b, k, p)$. This ends the proof of Theorem 1.1.

From Theorem 1.1 of this paper we get

Corollary 1.2. *If the non-decreasing functions a, b fulfil condition* (18) *and $l_1, l_2, \ldots, l_n \in \mathfrak{F}_\rho$, then*

$$(A, B) \in \bigcup_{i=1}^n T_{l_i}(a, b, 1, p) \text{ if and only if } (A, B) \in T_{l_1 + \cdots + l_n}(a, b, 1, p)$$

for arcs $A, B \in \widetilde{A}_p$.

If the condition

$$\lim_{A \ni x \to p} \frac{\ell(\widetilde{px})}{\rho(p, x)} = 1 \tag{32}$$

is fulfilled, then we say that the rectifiable arc $A \in E_0$ with the origin at the point $p \in E$ has the Archimedean property at the point p of the metric space (E, ρ).

Let A_p be the class of all rectifiable arcs having the Archimedean property at point $p \in E$. We note that all results presented in this paper are true for arbitrary arcs of the A_p class, because $A_p \subset \widetilde{A}_p$.

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