ADDITIVITY OF THE TANGENCY RELATION
OF RECTIFIABLE ARCS

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Abstract. In this paper the problem of the additivity of some tangency relation of sets for
the rectifiable arcs in the generalized metric spaces is considered. Some sufficient conditions
for the additivity of this relation are given.

Introduction

Let $E$ be an arbitrary non-empty set, and $E_0$ the family of all non-empty
subsets of set $E$. Let $l$ be a non-negative real function defined on the Cartesian
product $E_0 \times E_0$, and let $l_0$ be the function of the form:

$$l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for} \quad x, y \in E$$

$(1)$

We shall call pair $(E, l)$ the generalized metric space (see [1]). By some
assumptions relating to the function $l$, function $l_0$ defined by formula $(1)$ will
be the metric of set $E$.

Using $(1)$ we may define in the space $(E, l)$, similarly as in a metric space,
the following notions: sphere $S_l(p, r)$ and open ball $K_l(q, u)$

$$S_l(p, r) = \{x \in E : l_0(p, x) = r\} \quad \text{and} \quad K_l(q, u) = \{x \in E : l_0(q, x) < u\}$$

$(2)$

Let $a, b$ be arbitrary non-negative real functions defined in a certain right-
hand side neighbourhood of 0 such that

$$a(r) \xrightarrow{r \to 0^+} 0 \quad \text{and} \quad b(r) \xrightarrow{r \to 0^+} 0$$

$(3)$

We will denote by $S_l(p, r)_u$ (see [2, 3]) the $u$-neighbourhood of sphere $S_l(p, r)$
in space $(E, l)$ defined by the following formula:

$$S_l(p, r)_u = \left\{ \begin{array}{ll}
\bigcup_{q \in S_l(p, r)} K_l(q, u) & \text{for} \quad u > 0 \\
S_l(p, r) & \text{for} \quad u = 0
\end{array} \right.$$

$(4)$
We say that pair \((A, B)\) of sets \(A, B \in E_0\) is \((a, b)\)-clustered at point \(p\) of space \((E, l)\), if 0 is the cluster point of the set of all real numbers \(r > 0\) such that \(A \cap S_l(p, r)_{a(r)} \neq \emptyset\) and \(B \cap S_l(p, r)_{b(r)} \neq \emptyset\).

Let us define the following set (see \([1, 3, 4]\))

\[
T_{l}(a, b, k, p) = \{(A, B) : A, B \in E_0, \text{ pair } (A, B) \text{ is } (a, b)\text{-clustered at point } p \text{ of space } (E, l) \text{ and } \frac{1}{r^k}l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \xrightarrow{r \to 0^+} 0\}
\]  

(5)

If pair \((A, B) \in T_{l}(a, b, k, p)\), then we say that set \(A \in E_0\) is \((a, b)\)-tangent of order \(k > 0\) to set \(B \in E_0\) at point \(p\) of the generalizd metric space \((E, l)\).

Set \(T_{l}(a, b, k, p)\) defined by formula (5) is called the \((a, b)\)-tangency relation of order \(k\) of sets at point \(p\) in the generalized metric space \((E, l)\).

Let \(\rho\) be a metric of set \(E\) and let \(A, B\) be arbitrary sets of the family \(E_0\). Let us denote

\[
\rho(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}
\]

(6)

\[
d_{\rho}A = \sup\{\rho(x, y) : x, y \in A\}
\]

(7)

We shall denote by \(\mathcal{F}_{\rho}\) the class of all functions \(l\) fulfilling the conditions:

\[1^0] \quad l : E_0 \times E_0 \rightarrow [0, \infty),
\]

\[2^0] \quad \rho(A, B) \leq l(A, B) \leq d_{\rho}(A \cup B) \quad \text{for } A, B \in E_0.
\]

From equality (1) and from condition \(2^0\) it follows that

\[
l(\{x\}, \{y\}) = l_0(x, y) = \rho(x, y) \quad \text{for } l \in \mathcal{F}_{\rho} \text{ and } x, y \in E
\]

(8)

The above equality implies that any function \(l \in \mathcal{F}_{\rho}\) generates in set \(E\) the metric \(\rho\).

Let \(\tilde{A}_p\) be the class of the rectifiable arcs with the origin at point \(p \in E\) of the form (see \([5, 6, 7]\)):

\[
\tilde{A}_p = \{A \in E_0 : \lim_{A, x \rightarrow -p} \frac{\ell(\tilde{p}x)}{\rho(p, x)} = g < \infty\}
\]

(9)

where \(\ell(\tilde{p}x)\) denotes the length of the arc \(\tilde{p}x\) with ends \(p\) and \(x\).
Let \( l_1, l_2 \) be arbitrary functions belonging to the class \( \mathfrak{F}_\rho \). For these functions we define their sum as follows

\[
(l_1 + l_2)(A, B) = l_1(A, B) + l_2(A, B) \quad \text{for} \quad A, B \in E_0
\]

(10)

We say that tangency relation \( T_I(a, b, k, p) \) is additive in the class of functions \( \mathfrak{F}_\rho \) if

\[
(A, B) \in T_{l_1+l_2}(a, b, k, p) \quad \text{if and only if} \quad (A, B) \in T_{l_1}(a, b, k, p) \cup T_{l_2}(a, b, k, p)
\]

for \( l_1, l_2 \in \mathfrak{F}_\rho \) and \( A, B \in E_0 \).

In this paper the problem of the additivity of the tangency relation \( T_I(a, b, k, p) \) of the rectifiable arcs belonging to the class \( \mathfrak{A}_\rho \) in the spaces \( (E, l) \), for the functions \( l \in \mathfrak{F}_\rho \) is considered. Some sufficient conditions for the additivity of this tangency relation shall be given.

1. **On the additivity of the tangency relation of the arcs**

Let \( l_1, l_2 \) be the functions belonging to class \( \mathfrak{F}_\rho \) and \( E \) be any non-empty set.

**Lemma 1.1.** If functions \( l_1, l_2 \in \mathfrak{F}_\rho \), then

\[
S_{l_1+l_2}(p, r)_u = S_{\rho}(p, r/2)_{u/2}
\]

(11)

**Proof.** From equalities (8) and (10) we have

\[
K_{l_1+l_2}(p, r) = \{x \in E : (l_1 + l_2)((\{p\}, \{x\}) < r\}
\]

\[
= \{x \in E : l_1((\{p\}, \{x\}) + l_2((\{p\}, \{x\}) < r\} = \{x \in E : 2\rho(p, x) < r\}
\]

\[
= \{x \in E : \rho(p, x) < r/2\} = K_{\rho}(p, r/2).
\]

Therefore

\[
K_{l_1+l_2}(p, r) = K_{\rho}(p, r/2) \quad \text{for} \quad l_1, l_2 \in \mathfrak{F}_\rho
\]

(12)

Similarly

\[
S_{l_1+l_2}(p, r) = S_{\rho}(p, r/2) \quad \text{for} \quad l_1, l_2 \in \mathfrak{F}_\rho
\]

(13)

From definition (4) of the \( u \)-neighbourhood \( S_I(p, r)_u \) of sphere \( S_I(p, r) \), and from formulas (12) and (13) we obtain thesis (11) of the above lemma.

An immediate consequence of (13) is the following equality

\[
S_{l_1+\ldots+l_n}(p, r)_u = S_{\rho}(p, r/n)_{u/n} \quad \text{for} \quad l_1, \ldots, l_n \in \mathfrak{F}_\rho
\]

(14)

From Lemma 2.2 of the paper [2] (see also [8]) we get the following corollary:
Corollary 1.1. If function $a$ fulfils the condition
\[
\lim_{r \to 0^+} \frac{a(r)}{r} = 0
\]  
then for an arbitrary arc $A \in \tilde{A}_p$
\[
\lim_{r \to 0^+} \frac{1}{r} d_\rho(A \cap S_\rho(p, r/n)_{a(r)/n}) = 0
\]  
From condition (16) it follows immediately that
\[
\lim_{r \to 0^+} \frac{1}{r} d_\rho(A \cap S_\rho(p, r/2)_{a(r)/2}) = 0 \text{ for } A \in \tilde{A}_p
\]  
Now using these considerations we prove:

Theorem 1.1. If the non-decreasing functions $a, b$ fulfil the condition
\[
\lim_{r \to 0^+} \frac{a(r)}{r} = 0 \quad \text{and} \quad \lim_{r \to 0^+} \frac{b(r)}{r} = 0
\]  
and $l_1, l_2 \in \bar{S}_\rho$, then
\[(A, B) \in T_{l_1}(a, b, 1, p) \cup T_{l_2}(a, b, 1, p) \text{ if and only if } (A, B) \in T_{l_1 + l_2}(a, b, 1, p)\]  
for arcs $A, B \in \tilde{A}_p$.

Proof. We assume that $(A, B) \in T_{l_1}(a, b, 1, p) \cup T_{l_2}(a, b, 1, p)$. Hence in particular results that $(A, B) \in T_{l_1}(a, b, 1, p)$ for $A, B \in \tilde{A}_p$. From this we obtain
\[
\lim_{r \to 0^+} \frac{1}{r} l_1(A \cap S_\rho(p, r/2)_{a(r)/2}, B \cap S_\rho(p, r/2)_{b(r)/2}) = 0
\]  
Hence and from the fact that $l_1 \in \bar{S}_\rho$ we get
\[
\lim_{r \to 0^+} \frac{1}{r} \rho(A \cap S_\rho(p, r/2)_{a(r)/2}, B \cap S_\rho(p, r/2)_{b(r)/2}) = 0
\]  
The properties of functions $a, b$ imply the following inequality
\[
0 \leq \rho(A \cap S_\rho(p, r/2)_{a(r)}, B \cap S_\rho(p, r/2)_{b(r)}) \leq \rho(A \cap S_\rho(p, r/2)_{a(r)/2}, B \cap S_\rho(p, r/2)_{b(r)/2})
\]  
for arbitrary sets $A, B \in E_0$.
Thus, we can apply condition (20) and obtain
\[
\frac{1}{r}d_p((A \cap S_p(p, r/2)_{a(r)}) \cup (B \cap S_p(p, r/2)_{b(r)})) \xrightarrow{r \to 0^+} 0 
\]

(21)

Hence and from theorem on the compatibility of the tangency relations of sets of the class \(A_{p,1}^* \supset \tilde{A}_p\) (see Theorem 3 in paper [4]) we get

\[
\frac{1}{r}d_p((A \cap S_p(p, r/2)_{a(r)}) \cup (B \cap S_p(p, r/2)_{b(r)})) \xrightarrow{r \to 0^+} 0 
\]

(22)

As the following inequality

\[
0 \leq d_p((A \cap S_p(p, r/2)_{a(r)/2}) \cup (B \cap S_p(p, r/2)_{b(r)/2})) \\
\leq d_p((A \cap S_p(p, r/2)_{a(r)}) \cup (B \cap S_p(p, r/2)_{b(r)})) 
\]

is valid for any two sets \(A, B \in E_0\), then from formula (22) it follows

\[
\frac{1}{r}d_p((A \cap S_p(p, r/2)_{a(r)/2}) \cup (B \cap S_p(p, r/2)_{b(r)/2})) \xrightarrow{r \to 0^+} 0. 
\]

Hence we get

\[
\frac{1}{r}l_1(A \cap S_p(p, r/2)_{a(r)/2}, B \cap S_p(p, r/2)_{b(r)/2}) \xrightarrow{r \to 0^+} 0 
\]

(23)

for an arbitrary function \(l_1 \in \mathfrak{F}_p\).

From the fact that \(l_1, l_2 \in \mathfrak{F}_p\) and from Lemma 1.1 we obtain

\[
(l_1 + l_2)(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \\
= l_1(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \\
+ l_2(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \\
= l_1(A \cap S_p(p, r/2)_{a(r)/2}, B \cap S_p(p, r/2)_{b(r)/2}) \\
+ l_2(A \cap S_p(p, r/2)_{a(r)/2}, B \cap S_p(p, r/2)_{b(r)/2}) \\
\leq 2d_p((A \cap S_p(p, r/2)_{a(r)/2}) \cup (B \cap S_p(p, r/2)_{b(r)/2}))) \\
\leq 2d_p((A \cap S_p(p, r/2)_{a(r)/2}) + 2d_p((B \cap S_l(p, r/2)_{b(r)/2}) \\
+ 2p(A \cap S_p(p, r/2)_{a(r)/2}, B \cap S_p(p, r/2)_{b(r)/2}) \\
\leq 2d_p((A \cap S_p(p, r/2)_{a(r)/2}) + 2d_p((B \cap S_p(p, r/2)_{b(r)/2}) \\
+ 2l_1(A \cap S_p(p, r/2)_{a(r)/2}, B \cap S_p(p, r/2)_{b(r)/2}),
\]

whence
\[
\frac{1}{r}(l_1 + l_2)(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \\
\leq \frac{2}{r}d_p((A \cap S_{p}(p, r/2)_{a(r)/2}) + \frac{2}{r}d_p((B \cap S_{p}(p, r/2)_{b(r)/2}) \\
+ \frac{2}{r}l_1(A \cap S_{p}(p, r/2)_{a(r)/2}, B \cap S_{p}(p, r/2)_{b(r)/2})
\]

(24)

Formulas (23), (24) and Corollary 1.1 imply the relation

\[
\frac{1}{r}(l_1 + l_2)(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \longrightarrow 0
\]

(25)

valid for \(A, B \in \tilde{A}_p\) and \(l_1, l_2 \in \tilde{\mathcal{F}}_p\).

From the fact that \(A, B \in \tilde{A}_p\) it follows that pair of sets \((A, B)\) is \((a, b)\)-clustered at point \(p\) of space \((E, l_1 + l_2)\) for \(l_1, l_2 \in \tilde{\mathcal{F}}_p\). We conclude that \((A, B) \in T_{l_1+l_2}(a, b, 1, p)\).

Now we assume that \((A, B) \in T_{l_1+l_2}(a, b, 1, p)\) for \(A, B \in \tilde{A}_p\). Hence we have

\[
\frac{1}{r}(l_1 + l_2)(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \longrightarrow 0.
\]

From the above relation and from definition (10) it follows that

\[
\frac{1}{r}l_1(A \cap S_{l_1+l_2}(p, r)_{a(r)}, B \cap S_{l_1+l_2}(p, r)_{b(r)}) \longrightarrow 0
\]

(26)

Thus, we apply Lemma 1.1 and (26) we get

\[
\frac{1}{r}l_1(A \cap S_{p}(p, r/2)_{a(r)/2}, B \cap S_{p}(p, r/2)_{b(r)/2}) \longrightarrow 0.
\]

The assumption: \(l_1 \in \tilde{\mathcal{F}}_p\) yields the following formula

\[
\frac{1}{r}\rho(A \cap S_{p}(p, r/2)_{a(r)/2}, B \cap S_{p}(p, r/2)_{b(r)/2}) \longrightarrow 0
\]

(27)

As inequality

\[
0 \leq \rho(A \cap S_{p}(p, r/2)_{a(r)}, B \cap S_{p}(p, r/2)_{b(r)}) \\
\leq \rho(A \cap S_{p}(p, r/2)_{a(r)/2}, B \cap S_{p}(p, r/2)_{b(r)/2})
\]

is valid for arbitrary sets \(A, B \in E_0\), then from formula (27) it follows that

\[
\frac{1}{r}\rho(A \cap S_{p}(p, r/2)_{a(r)}, B \cap S_{p}(p, r/2)_{b(r)}) \longrightarrow 0
\]

(28)
Hence we get
\[
\frac{1}{r}d_p((A \cap S_\rho(p, r/2))_{a(r)}) \cup (B \cap S_\rho(p, r/2))_{b(r)}) \xrightarrow{r \to 0^+} 0 \tag{29}
\]
As the following inequality
\[
0 \leq d_p((A \cap S_\rho(p, r/2))_{a(r/2)}) \cup (B \cap S_\rho(p, r/2))_{b(r/2)})
\leq d_p((A \cap S_t(p, r/2))_{a(r)}) \cup (B \cap S_t(p, r/2))_{b(r)})
\]
is fulfilled for \( A, B \in E_0 \), then from formula (29) it follows
\[
\frac{1}{r}d_p((A \cap S_\rho(p, r/2))_{a(r/2)}) \cup (B \cap S_\rho(p, r/2))_{b(r/2)}) \xrightarrow{r \to 0^+} 0 \tag{30}
\]
Applying assumption: \( l_1 \in \mathfrak{F}_\rho \) we obtain
\[
\frac{1}{r}l_1(A \cap S_\rho(p, r/2))_{a(t)}, B \cap S_\rho(p, r/2))_{b(t)}) \xrightarrow{r \to 0^+} 0,
\]
i.e.
\[
\frac{1}{l}l_1(A \cap S_\rho(p, t))_{a(t)}, B \cap S_\rho(p, t))_{b(t)}) \xrightarrow{t \to 0^+} 0 \tag{31}
\]
From the fact that pair of arcs \((A, B)\) is \((a, b)\)-clustered at point \( p \) of space \((E, l_1)\) and from condition (31) it follows that \((A, B) \in T_{l_1}(a, b, 1, p)\). Therefore \((A, B) \in T_{l_1}(a, b, k, p) \cup T_{l_2}(a, b, k, p)\). This ends the proof of Theorem 1.1.

From Theorem 1.1 of this paper we get

**Corollary 1.2.** If the non-decreasing functions \( a, b \) fulfil condition (18) and \( l_1, l_2, \ldots, l_n \in \mathfrak{F}_\rho \), then
\[
(A, B) \in \bigcup_{i=1}^n T_{l_i}(a, b, 1, p) \text{ if and only if } (A, B) \in T_{l_1 + \ldots + l_n}(a, b, 1, p)
\]
for arcs \( A, B \in \tilde{A}_p \).

If the condition
\[
\lim_{A \xrightarrow{p \rho} p} \frac{\ell(p, x)}{\ell(p, x)} = 1 \tag{32}
\]
is fulfilled, then we say that the rectifiable arc \( A \in E_0 \) with the origin at the point \( p \in E \) has the Archimedean property at the point \( p \) of the metric space \((E, \rho)\).

Let \( A_p \) be the class of all rectifiable arcs having the Archimedean property at point \( p \in E \). We note that all results presented in this paper are true for arbitrary arcs of the \( A_p \) class, because \( A_p \subset \tilde{A}_p \).
References