APPLICATION OF THE RAYLEIGH-RITZ METHOD FOR SOLVING FRACTIONAL OSCILLATOR EQUATION

Tomasz Błaszczyk

Institute of Mathematics, Czestochowa University of Technology, Poland
tomblaszczyk@gmail.com

Abstract. In this work a fractional oscillator equation is considered. This type of equation includes a composition of left and right fractional derivatives. A scheme based on the variational Rayleigh-Ritz method is proposed to obtain a numerical solution of the problem.

Introduction

Fractional oscillator equation is a type of equation which includes a composition of left and right fractional derivatives. This type of equations appears in theoretical fractional mechanics while using the minimum action principle and fractional integration by parts rule. Riewe [1, 2] was the first author who used this method in derivation of fractional differential equations in mechanics. Later sequential Lagrangian and Hamiltonian approaches to the problem were proposed (see for example, [3-10]). Using the fixed point theorems [11-13] one can obtain analytical results. Unfortunately, this solution is represented by series of alternately left and right fractional integrals and therefore is difficult in any practical calculations. In order to generate analytical solution Klimek in [14] shows an application of the Mellin transform, but this solution is represented by complicated series of special functions.

Analytical results obtained so far are inspiration to look for approximate solutions. In [15] some approximate solutions based on Fractional Power Series, for a class of Fractional Optimal Control problems is presented. In this paper a numerical scheme based on Rayleigh-Ritz method [16, 17] for fractional oscillator equation is proposed.

1. Basic definitions and formulation of the problem

We recall some definitions of the fractional operators [18]:

– left fractional Riemann-Liouville integral:

$$I_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \qquad t > 0 \tag{1}$$

– right fractional Riemann-Liouville integral:

$$I_{b-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds \quad t < b \tag{2}$$

where $\alpha \in R_+$. Using the above fractional integrals we define fractional derivatives. The left fractional Riemann-Liouville derivative looks as follows (we have denoted the classical derivative as $D := \frac{d}{dt}$) [18]:

$$D_{0+}^{\alpha} f(t) := D^n I_{0+}^{n-\alpha} f(t) \tag{3}$$

and for the right fractional Riemann-Liouville derivative we have [18]:

$$D_{b-}^{\alpha} f(t) := (-D)^n I_{b-}^{n-\alpha} f(t) \tag{4}$$

where $n = [\alpha] + 1$ ($[\alpha]$ is the integer part of α). Now we define the right fractional Caputo derivative [18]:

$${}^{C}D_{b-}^{\alpha} f(t) := D_{b-}^{\alpha} f(t) - \sum_{j=1}^{n-1} \frac{(b-t)^{j-\alpha}}{\Gamma(j-\alpha+1)} D^j f(b) \tag{5}$$

We shall consider fractional oscillator equation of the form:

$${}^{C}D_{1-}^{\alpha} D_{0+}^{\alpha} f(t) + \lambda f(t) = \mu g(t), \qquad t \in [0,1],\ \alpha \in (0,1) \tag{6}$$

where $\lambda, \mu \in R$ and f is a continuous function, fulfilling the conditions:

$$f(0) = f(1) = 0 \tag{7}$$

The analytical solution of (6) in the special case for $g(t) = 0$ was obtained by Klimek in [13, 14].
To apply the Rayleigh-Ritz method for equation (6) we consider the functional of the form:

$$I(f) = \int_0^1 \left[\frac{1}{2} \left(D_{0+}^{\alpha} f \right)^2 + \frac{\lambda}{2} f^2 - \mu \cdot f \cdot g \right] dt \tag{8}$$

2. Numerical technique

In this section we present numerical scheme based on the Rayleigh-Ritz method. Let us assume that the solution of equation (6) with conditions (7), can be written as:

$$f_m(t) = \sum_{k=1}^{m} a_k N_k(t) \tag{9}$$

where a_k are unknown constant coefficients to be determined, and $N_k(t)$ are test functions fulfilling conditions (7). We assume that functions $N_1(t),\ldots,N_m(t)$ have the left fractional Riemann-Liouville derivatives.
Substituting (9) into (8), we obtain:

$$\begin{aligned} I(a_1,\ldots,a_m) = &\int_0^1 \left[\frac{1}{2}\left(D_{0+}^{\alpha}\left(\sum_{k=1}^{m} a_k N_k(t) \right)\right)^2 + \frac{\lambda}{2}\left(\sum_{k=1}^{m} a_k N_k(t) \right)^2 \right] dt \\ &- \int_0^1 \mu \cdot g(t) \cdot \sum_{k=1}^{m} a_k N_k(t) dt \end{aligned} \tag{10}$$

Minimizing functional I leads to the system of equations:

$$\begin{cases} \dfrac{\partial}{\partial a_1}\left(\displaystyle\int_0^1 \left[\frac{1}{2}\left(D_{0+}^{\alpha}\left(\sum_{k=1}^{m} a_k N_k(t) \right)\right)^2 + \frac{\lambda}{2}\left(\sum_{k=1}^{m} a_k N_k(t) \right)^2 \right] dt \right) = \dfrac{\partial}{\partial a_1}\left(\displaystyle\int_0^1 \mu \cdot g(t) \cdot \sum_{k=1}^{m} a_k N_k(t) dt \right) \\ \dfrac{\partial}{\partial a_2}\left(\displaystyle\int_0^1 \left[\frac{1}{2}\left(D_{0+}^{\alpha}\left(\sum_{k=1}^{m} a_k N_k(t) \right)\right)^2 + \frac{\lambda}{2}\left(\sum_{k=1}^{m} a_k N_k(t) \right)^2 \right] dt \right) = \dfrac{\partial}{\partial a_1}\left(\displaystyle\int_0^1 \mu \cdot g(t) \cdot \sum_{k=1}^{m} a_k N_k(t) dt \right) \\ \vdots \\ \dfrac{\partial}{\partial a_m}\left(\displaystyle\int_0^1 \left[\frac{1}{2}\left(D_{0+}^{\alpha}\left(\sum_{k=1}^{m} a_k N_k(t) \right)\right)^2 + \frac{\lambda}{2}\left(\sum_{k=1}^{m} a_k N_k(t) \right)^2 \right] dt \right) = \dfrac{\partial}{\partial a_1}\left(\displaystyle\int_0^1 \mu \cdot g(t) \cdot \sum_{k=1}^{m} a_k N_k(t) dt \right) \end{cases} \tag{11}$$

with unknowns a_k, $k = 1,\ldots,m$.
Calculating derivatives and doing some algebraic manipulations we obtain the following system of linear equations:

$$\begin{cases} \int_0^1 p_1 \sum_{k=1}^m a_k p_k dt + \lambda \int_0^1 N_1(t) \sum_{k=1}^m a_k N_k(t) dt = \mu \int_0^1 N_1(t) g(t) dt \\ \int_0^1 p_2 \sum_{k=1}^m a_k p_k dt + \lambda \int_0^1 N_2(t) \sum_{k=1}^m a_k N_k(t) dt = \mu \int_0^1 N_2(t) g(t) dt \\ \vdots \\ \int_0^1 p_m \sum_{k=1}^m a_k p_k dt + \lambda \int_0^1 N_m(t) \sum_{k=1}^m a_k N_k(t) dt = \mu \int_0^1 N_m(t) g(t) dt \end{cases} \tag{12}$$

where $p_k = D_{0+}^{\alpha} N_k(t)$.
The system (12) can be written in the matrix form as:

$$B \cdot X_a = C \tag{13}$$

where:

$$B = \begin{bmatrix} \int_0^1 \left(p_1^2 + \lambda N_1^2(t) \right) dt & \int_0^1 \left(p_1 p_2 + \lambda N_1(t) N_2(t) \right) dt & \cdots & \int_0^1 \left(p_1 p_m + \lambda N_1(t) N_m(t) \right) dt \\ \int_0^1 \left(p_1 p_2 + \lambda N_1(t) N_2(t) \right) dt & \int_0^1 \left(p_2^2 + \lambda N_2^2(t) \right) dt & \cdots & \int_0^1 \left(p_2 p_m + \lambda N_2(t) N_m(t) \right) dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 \left(p_1 p_m + \lambda N_1(t) N_m(t) \right) dt & \int_0^1 \left(p_2 p_m + \lambda N_2(t) N_m(t) \right) dt & \cdots & \int_0^1 \left(p_m^2 + \lambda N_m^2(t) \right) dt \end{bmatrix}$$

$$X_a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \qquad C = \begin{bmatrix} \mu \int_0^1 N_1(t) g(t) dt \\ \mu \int_0^1 N_2(t) g(t) dt \\ \vdots \\ \mu \int_0^1 N_m(t) g(t) dt \end{bmatrix} \tag{14}$$

We get the system of m - linear equations, from which we can obtain coefficients a_k, $k = 1, \ldots, m$.

3. Numerical example

Now, we present the results of calculation obtained by our numerical method. As an example we study the fractional oscillator equation in the form:

$$^{C}D^{\alpha}_{1-}D^{\alpha}_{0+}f(t)+\pi^2 f(t)=2\pi^2\sin(\pi t),\qquad t\in[0,1],\ \alpha\in(0,1) \tag{15}$$

with conditions (7).
If order $\alpha\to 0^+$ then we get in the limit equation:

$$f(t)+\pi^2 f(t)=2\pi^2\sin(\pi t) \tag{16}$$

with the solution:

$$f(t)=\frac{2\pi^2\sin(\pi t)}{1+\pi^2} \tag{17}$$

If order $\alpha\to 1^-$ then we get the following ordinary differential equation:

$$-D^2 f(t)+\pi^2 f(t)=2\pi^2\sin(\pi t) \tag{18}$$

with the solution:

$$f(t)=\sin(\pi t) \tag{19}$$

To obtain the numerical solution of equation (15) obeying conditions (7) we assume that the solution $f_{m,\alpha}$ has the form:

$$f_{m,\alpha}(t)=\sum_{k=1}^{m}a_k N_k(t)=\sum_{k=1}^{m}a_k t^{k+\alpha}\left(1-t^{\alpha}\right) \tag{20}$$

Let us observe that functions N_k also fulfill conditions (7). Moreover, they have the left fractional Riemann-Liouville derivatives D^{α}_{0+} given as:

$$D^{\alpha}_{0+}\left[t^{k+\alpha}\left(1-t^{\alpha}\right)\right]=t^{k}\left(\frac{\Gamma(k+1+\alpha)}{\Gamma(k+1)}-\frac{\Gamma(k+1+2\alpha)}{\Gamma(k+1+\alpha)}t^{\alpha}\right) \tag{21}$$

We calculated some examples for different values of α to show graphically how the numerical solutions behave. Approximate solutions of equation (15) and analytical solutions of equations (16), (18), are presented on Figure 1.

Analysing behaviour of the solutions we observe that, if *m* grows and $\alpha \to 0^+$ then $f_{m,\alpha}$ tend to solution (17), while if *m* grows and $\alpha \to 1^-$ then $f_{m,\alpha}$ tend to solution (19).

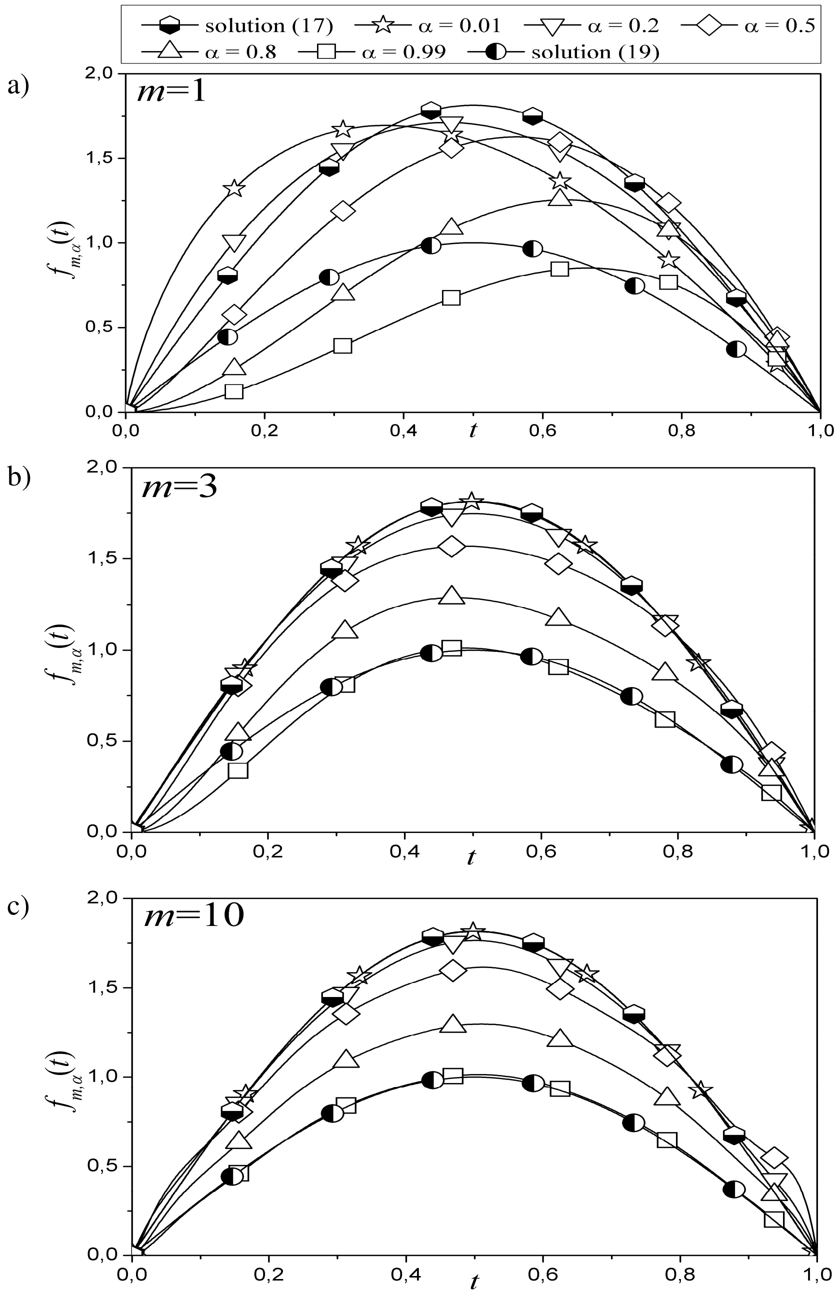

Fig. 1. Approximate solutions of equation (15) for: (a) $m = 1$, (b) $m = 3$, (c) $m = 10$ and analytical solutions (17), (19)

Conclusions

In this work a fractional oscillator equation was considered. This type of equation includes a composition of the left and the right fractional derivatives. The analytical solution of such an equation is represented by series of alternately left and right fractional integrals and therefore is difficult to apply in any practical calculations. Numerical solution is an alternative approach to the analytical one. In this study the scheme based on the variational Rayleigh-Ritz method was presented to obtain a numerical solution of the fractional oscillator equation. Analysing solutions presented by the graphs we observe that the solutions of fractional oscillator equation (15) are located between analytical solutions of equatios (16) and (18) respectively. Our results show that the solution of the fractional oscillator equation approaches the solution of the classical ordinary differential equation when order $\alpha \to 1^{-}$.

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