THE POLYNOMIAL TENSOR INTERPOLATION.
ARITHMETICAL CASE

Grzegorz Biernat, Anita Ciekot
Institute of Mathematics, Czestochowa University of Technology, Poland, ciekot@imi.pcz.pl

Abstract. In this paper the tensor interpolation by polynomials of several variables is considered. The effective formulas for polynomial coefficients for arithmetical case were obtained.

Introduction

The formulas of tensor interpolation by polynomials of several variables are unknow in the interpolation methods [1]. Using the Kronecker tensor product of matrices [2, 3] the polynomial tensor interpolation formula was given in the previous articles [4, 5]. In this paper we considere the arithmetical case of the nodes matrix.

The Polynomial Arithmetical Tensor Interpolation

The coefficients matrix \([ a_{h_{1}...h_{k}} ] \) of the polynomial arithmetic tensor interpolation

\[
W(X_1,...,X_k) = \sum_{0 \leq h_1 \leq p_1, ... , 0 \leq h_k \leq p_k} a_{h_{1}...h_{k}} X_1^{h_{1}} \cdots X_k^{h_{k}}
\]

are unknow.

The results matrix \([ w_{i_1} \cdots w_{i_k} ] \) and the nodes matrix \([ X_{i_1} \cdots X_{i_k} ] \) are know and

\[
X_{i_1} = X_{1_{i_1}} + i_1 \Delta_1, \quad 0 \leq i_1 \leq p_1
\]

\[
X_{i_2} = X_{2_{i_2}} + i_2 \Delta_2, \quad 0 \leq i_2 \leq p_2
\]

\[.................................\]

\[
X_{i_k} = X_{k_{i_k}} + i_k \Delta_k, \quad 0 \leq i_k \leq p_k
\]
where \( \Delta_i \) is a common difference of \( X_{l_i} \) sequence.

**Fact 1.** For the arithmetical sequence \( X_i = X_0 + i\Delta \) \( (i = 0, 1, \ldots, p) \) we have \([5]\)

\[
P_i = (X_p - X_i) \cdot \cdots \cdot (X_{i+1} - X_i) \cdot (X_i - X_{i-1}) \cdot \cdots \cdot (X_1 - X_0) = (p-i)\Delta \cdot \cdots \cdot \Delta \cdot 2\Delta \cdots \cdot i\Delta = (p-i)!\Delta^{p-i} i! = (p-i)!i!\Delta^p
\]

**Lemma 1.** For integers \( 1 \leq q \leq l \leq p \) we have

\[
\sum_{t \leq \kappa_1 < \kappa_2 \ldots < \kappa_q \leq p} \tau_q(\kappa_1, \ldots, \kappa_q) = \binom{p-q}{l-q} \tau_q(1,2,\ldots,p)
\]

where \( \tau_q \) is the symmetric polynomial of \( q \) - order.

**Proof.** For the numbers \( 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_q \leq p \) from sequence \( (\kappa_1, \kappa_2, \ldots, \kappa_q) \) we have \( p-l \) remaining values disposed to \( q-l \) places. So, each component \( \alpha_1 \alpha_2 \ldots \alpha_q \) of symmetric polynomial \( \tau_q(1,2,\ldots,p) \) is repeated in the left hand sum \( \binom{p-q}{l-q} \) times.

**Lemma 2.** For integers \( 1 \leq q \leq p \) we have

\[
\tau_q(1,2,\ldots,i,\ldots,p) = \tau_q(1,2,\ldots,p) - i\tau_{q-1}(1,2,\ldots,i,\ldots,p)
\]

where the symbol \( \hat{i} \) means omitting the variable \( i \).

**Corollary 1.** For integers \( 1 \leq q \leq p \) we obtain

\[
\tau_q(1,2,\ldots,i,\ldots,p) = \tau_q(1,2,\ldots,p) - i\tau_{q-1}(1,2,\ldots,p) + \\
i^2\tau_{q-2}(1,2,\ldots,p) + \cdots + (-1)^i i^q \tau_0
\]

**Fact 2.** For arithmetical sequence \( X_i = X_0 + i\Delta \) \( (i = 0, 1, \ldots, p) \) we have
\[ \tau_i(X_0, X_1, \ldots, \hat{X}_i, \ldots, X_p) = \sum_{q=0}^{p} \left( p - q \right) \left[ \sum_{l=0}^{q} (-1)^i \tau_{q-i}(1, 2, \ldots, p) \right] \Delta^q X_{0-l}^q \]

In this formula we assume that \( 0^0 = 1 \).

**Proof.** In fact, according to lemma 1 and corollary 1 we obtain

\[ \tau_i(X_0, X_1, \ldots, \hat{X}_i, \ldots, X_p) = \sum_{k_j \in_i = 1, 2, \ldots, p} X_{k_j} \ldots X_{k_i} = \]

\[ \sum_{k_j \in_i = 1, 2, \ldots, p} (X_0 + k_j \Delta) \ldots (X_0 + k_i \Delta) = \]

\[ \sum_{k_j \in_i = 1, 2, \ldots, p} (X_0^l + \tau_i(k_1 \ldots k_j) \Delta X_{0-1}^l + \]

\[ \tau_2(k_1 \ldots k_j) \Delta^2 X_{0-2}^l + \ldots + \tau_i(k_1 \ldots k_j) \Delta X_{0-l}^i = \]

\[ \left( \begin{array}{c} p \\ l \end{array} \right) X_0^l + \left( \sum_{k_j \in_i = 1, 2, \ldots, p} \tau_i(k_1 \ldots k_i) \right) \Delta X_{0-l}^i + \]

\[ \left( \sum_{k_j \in_i = 1, 2, \ldots, p} \tau_2(k_1 \ldots k_i) \right) \Delta^2 X_{0-2}^l + \ldots + \]

\[ \left( \sum_{k_j \in_i = 1, 2, \ldots, p} \tau_i(k_1 \ldots k_i) \right) \Delta^l = \]

\[ \left( \begin{array}{c} p \\ l \end{array} \right) X_0^l + \left( \begin{array}{c} p-1 \\ l-1 \end{array} \right) \tau_i(1, 2, \ldots, \hat{i}, \ldots, p) \Delta X_{0-l}^i + \]

\[ \left( \begin{array}{c} p-2 \\ l-2 \end{array} \right) \tau_i(1, 2, \ldots, \hat{i}, \ldots, p) \Delta^2 X_{0-2}^l + \ldots + \tau_i(1, 2, \ldots, \hat{i}, \ldots, p) \Delta^l = \]

\[ \left( \begin{array}{c} p \\ l \end{array} \right) X_0^l + \sum_{q=1}^{p} \left( \begin{array}{c} p-q \\ l-q \end{array} \right) \left( \sum_{i=0}^{q} (-1)^i \tau_{q-i}(1, 2, \ldots, p) \right) \Delta^q X_{0-l}^q \]
According to above facts we obtain

**Corollary 2.** In the formulas of polynomial coefficients of arithmetical tensor interpolation [5] we have

\[
\tau_{p_i-h} (X_{10}, \ldots, \hat{X}_{i_h}, \ldots, X_{ip}) = \sum_{q=0}^{n-1} \left( \frac{p_i - q_i}{p_i - j_i - q_i} \right) \sum_{j=0}^{q} (-1)^j q_i^{q-j} \tau_{q-i} (1, 2, \ldots, p_i) \Delta^n X_{10}^{p_i-n-h-q}
\]

\[
\tau_{p_k-h} (X_{10}, \ldots, \hat{X}_{i_k}, \ldots, X_{ip}) = \sum_{q=0}^{n-1} \left( \frac{p_k - q_k}{p_k - j_k - q_k} \right) \sum_{j=0}^{q} (-1)^j q_k^{q-j} \tau_{q-k} (1, 2, \ldots, p_k) \Delta^n X_{10}^{p_k-n-k-q}
\]

and

\[
\Pi_{i_h} = (p_i - i_i) ! i_i ! \Delta^{p_i}
\]

\[
\Pi_{k_h} = (p_k - i_k) ! i_k ! \Delta^{p_k}
\]

so

\[
a_{j_1,\ldots,j_k} = (-1)^{j_1+\ldots+j_k} \Delta^{p_i}_{j_1,\ldots,j_k} \left( \sum_{0 \leq q_1 \leq p_i, \ldots, 0 \leq q_k \leq p_k} (-1)^{j_1+\ldots+j_k} \tau_{q_1-q_1} (1, 2, \ldots, p_i) \Delta^{q_1}_1 X_{10}^{p_i-n-j_1-q_1}
\]

\[
\sum_{q=0}^{n-1} \left( \frac{p_i - q_i}{p_i - j_i - q_i} \right) \sum_{j=0}^{q} (-1)^j q_i^{q-j} \tau_{q-i} (1, 2, \ldots, p_i) \Delta^n X_{10}^{p_i-n-h-q}
\]

\[
\sum_{q=0}^{n-1} \left( \frac{p_k - q_k}{p_k - j_k - q_k} \right) \sum_{j=0}^{q} (-1)^j q_k^{q-j} \tau_{q-k} (1, 2, \ldots, p_k) \Delta^n X_{10}^{p_k-n-k-q}
\]

In above formulas we assume that \(0^0 = 1\).

**Conclusion**

In this article the effective formulas for the polynomial coefficients of arithmetical tensor interpolation were obtained. Only the values of symmetric polynomials for natural numbers \(\tau_{k} (1, 2, \ldots, p)\) are necessary.
References