OPTIMAL LOCATION OF SENSORS FOR ESTIMATION OF CAST IRON LATENT HEAT

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Abstract. The thermal processes proceeding in a system casting-mould are considered. The casting is made from cast iron and the austenite and eutectic latent heats should be identified. To estimate these parameters the knowledge of temperature history at the points selected from the domain considered is necessary. The location of sensors should assure the best conditions of identification process. In the paper the method of optimum location of sensors basing on the D-optimality criterion is presented.

1. Formulation of the problem

A system casting-mould-environment is considered. Temperature field in casting domain is described by equation [1, 2]

$$x \in \Omega: \quad C(T)\frac{\partial T(x, t)}{\partial t} = \lambda \nabla^2 T(x, t) \tag{1}$$

where $C(T)$ is the substitute thermal capacity of cast iron, λ is the thermal conductivity, T, x, t denote the temperature, geometrical co-ordinates and time.

The following approximation of substitute thermal capacity is taken into account

$$C(T) = \begin{cases} c_L, & T > T_L \\ \dfrac{c_L + c_S}{2} + \dfrac{Q_{aus}}{T_L - T_E}, & T_E < T \le T_L \\ \dfrac{c_L + c_S}{2} + \dfrac{Q_{eu}}{T_E - T_S}, & T_S < T \le T_E \\ c_S, & T \le T_S \end{cases} \tag{2}$$

where T_L, T_S are the liquidus and solidus temperatures, respectively, T_E is the temperature corresponding to the beginning of eutectic crystallization, Q_{aus}, Q_{eu} are the latent heats connected with the austenite and eutectic phases evolution, c_L , c_S are constant volumetric specific heats of molten metal and solid one, respectively.

The temperature field in mould sub-domain is described by equation

$$x \in \Omega_m: \quad c_m \frac{\partial T_m(x, t)}{\partial t} = \lambda_m \nabla^2 T_m(x, t) \tag{3}$$

where c_m is the mould volumetric specific heat, λ_m is the mould thermal conductivity.

On the contact surface between casting and mould the continuity condition in the form

$$x \in \Gamma_c: \begin{cases} -\lambda\, \mathbf{n} \cdot \nabla T(x, t) = -\lambda_m \mathbf{n} \cdot \nabla T_m(x, t) \\ T(x, t) = T_m(x, t) \end{cases} \tag{4}$$

can be accepted.

On the external surface of the system the Robin condition

$$x \in \Gamma_0: \quad -\lambda_m \mathbf{n} \cdot \nabla T_m(x, t) = \alpha\left[T_m(x, t) - T_a\right] \tag{5}$$

is given (α is the heat transfer coefficient, T_a is the ambient temperature).

For time $t = 0$ the initial condition

$$t = 0: \quad T(x, 0) = T_0(x), \quad T_m(x, 0) = T_{m0}(x) \tag{6}$$

is also known.

It is assumed that the aim of experiments is to determine the latent heats Q_{aus}, Q_{eu} of casting material and in order to find the optimal location of sensors the D-optimality criterion [3, 4] is taken into account.

2. Optimal sensors location

The model above formulated contains two unknown parameters Q_{aus}, Q_{eu} which will be reconstructed on the basis of observations. Let us assume that the approximate values of Q_{aus}^0, Q_{eu}^0 are available e.g. from preliminary experiments. Our goal is to determine the optimal sensors location in order to maximize the expected accuracy of parameters identification which will be found using the data generated in new experiments.

In Figure 1 the domain considered and its discretization is shown. Let $x^1 = (x_1^1, x_2^1)$, $x^2 = (x_1^2, x_2^2)$, ... , $x^M = (x_1^M, x_2^M)$ are the points from the casting sub-domain which are taken into account as the possible sensors location (Fig. 1). The design problem consists in the selection of the best positions of sensors under the assumption that only two sensors will be taken into account (it corresponds to the number of estimated parameters).

Let us introduce the sensitivity coefficients

$$Z_{l1}^{f} = \frac{\partial T\left(x^{l},\, t^{f},\, Q_{aus}^{0},\, Q_{eu}^{0}\right)}{\partial Q_{aus}^{0}}, \quad Z_{l2}^{f} = \frac{\partial T\left(x^{l},\, t^{f},\, Q_{aus}^{0},\, Q_{eu}^{0}\right)}{\partial Q_{eu}^{0}} \tag{7}$$

where l denotes number of node ($l = 1,2,...,M$) and t^f is the moment of time ($f =$ $= 1,2, ...,F$). On the basis of (7) the sensitivity matrix is constructed

$$\mathbf{Z}\left(x^{i}, x^{j}\right) = \begin{bmatrix} Z_{i1}^{1} & Z_{i2}^{1} \\ ... & ... \\ Z_{i1}^{F} & Z_{i2}^{F} \\ Z_{j1}^{1} & Z_{j2}^{1} \\ ... & ... \\ Z_{j1}^{F} & Z_{j2}^{F} \end{bmatrix} \tag{8}$$

where x^i, x^j are the pair of nodes, at the same time $i, j = 1, 2, ..., M$, $i \neq j$.

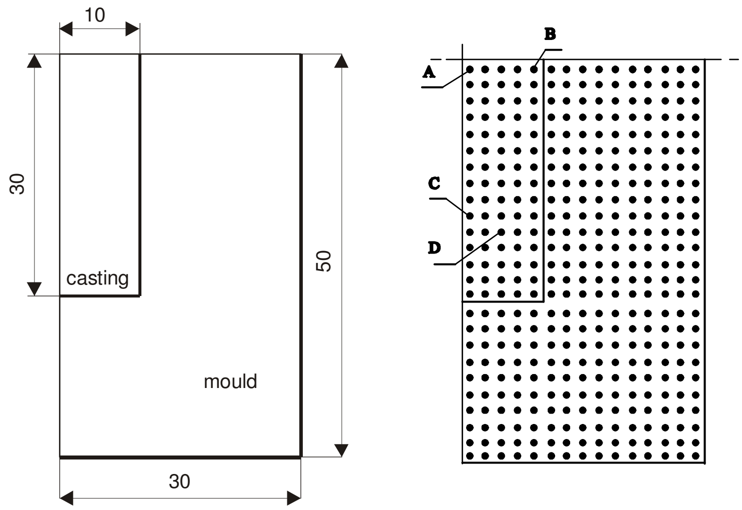

Fig. 1. Domain considered and its discretization

D-optimality criterion used in the design of sensors location is the following [3, 4]

$$\det \mathbf{M}\left(x^{i*}, x^{j*}\right) = \max_{\left(x^{i}, x^{j}\right)} \det \mathbf{M}\left(x^{i}, x^{j}\right) \tag{9}$$

where

$$\mathbf{M}\left(x^{i}, x^{j}\right) = \mathbf{Z}^{\mathrm{T}}\left(x^{i}, x^{j}\right) \mathbf{Z}\left(x^{i}, x^{j}\right) \tag{10}$$

It is easy to check that

$$\mathbf{M}\left(x^{i}, x^{j}\right)=\begin{bmatrix} \sum_{f=1}^{F}\left(Z_{i1}^{f}\right)^{2}+\left(Z_{j1}^{f}\right)^{2} & \sum_{f=1}^{F} Z_{i1}^{f} Z_{i2}^{f}+Z_{j1}^{f} Z_{j2}^{f} \\ \sum_{f=1}^{F} Z_{i1}^{f} Z_{i2}^{f}+Z_{j1}^{f} Z_{j2}^{f} & \sum_{f=1}^{F}\left(Z_{i2}^{f}\right)^{2}+\left(Z_{j2}^{f}\right)^{2} \end{bmatrix} \tag{11}$$

The nodes (x^{i*}, x^{j*}) being the solution of optimum problem (9) correspond to the best sensors location in the case of simultaneous identification of parameters Q_{aus} and Q_{eu}.

3. Sensitivity coefficients

Elementary step in the design of optimal sensor locations is to use an effective procedure for the computations of sensitivity coefficients. One of the method is the direct differentiation of governing equations with respect to the identified parameters [2, 5-8]. So, differentiation of equations (1)-(6) with respect to $p_1 = Q_{aus}$, $p_2 = Q_{eu}$ leads to the following additional boundary-initial problems

$$\begin{aligned} & x\in\Omega: \frac{\partial C(T)}{\partial p_e}\frac{\partial T(x,t)}{\partial t}+C(T)\frac{\partial}{\partial p_e}\left(\frac{\partial T(x,t)}{\partial t}\right)=\lambda\frac{\partial \nabla^2 T(x,t)}{\partial p_e} \\ & x\in\Omega_m: c_m\frac{\partial}{\partial p_e}\left(\frac{\partial T_m(x,t)}{\partial t}\right)=\lambda_m\frac{\partial \nabla^2 T_m(x,t)}{\partial p_e} \\ & x\in\Gamma_c: \begin{cases} -\lambda\mathbf{n}\cdot\dfrac{\partial \nabla T(x,t)}{\partial p_e}=-\lambda_m\mathbf{n}\cdot\dfrac{\partial \nabla T_m(x,t)}{\partial p_e} \\ \dfrac{\partial T(x,t)}{\partial p_e}=\dfrac{\partial T_m(x,t)}{\partial p_e} \end{cases} \\ & x\in\Gamma_0: -\lambda_m\mathbf{n}\cdot\frac{\partial \nabla T_m(x,t)}{\partial p_e}=\alpha\frac{\partial T_m(x,t)}{\partial p_e} \\ & t=0: \frac{\partial T(x,0)}{\partial p_e}=0, \quad \frac{\partial T_m(x,0)}{\partial p_e}=0 \end{aligned} \tag{12}$$

or

$$
\begin{aligned}
&x \in \Omega : C(T)\frac{\partial Z_e(x,t)}{\partial t} = \lambda \nabla^2 Z_e(x,t) - \frac{\partial C(T)}{\partial p_e}\frac{\partial T(x,t)}{\partial t}\\
&x \in \Omega_m : c_m \frac{\partial Z_{me}(x,t)}{\partial t} = \lambda_m \nabla^2 Z_{me}(x,t)\\
&x \in \Gamma_c : \begin{cases} -\lambda \mathbf{n}\cdot\nabla Z_e(x,t) = -\lambda_m \mathbf{n}\cdot\nabla Z_{me}(x,t)\\ Z_e(x,t) = Z_{me}(x,t)\end{cases}\\
&x \in \Gamma_0 : \ -\lambda_m \mathbf{n}\cdot\nabla Z_{me}(x,t) = \alpha Z_{me}(x,t)\\
&t = 0: \quad Z_e(x,0) = 0 \ , \ Z_{me}(x,0) = 0
\end{aligned}
\tag{13}
$$

where

$$
Z_e(x,t) = \frac{\partial T(x,t)}{\partial p_e}, \quad Z_{me}(x,t) = \frac{\partial T_m(x,t)}{\partial p_e} \tag{14}
$$

Summing up, in order to construct the sensitivity matrix (8) the basic problem described by equations (1)-(6) and the sensitivity problems (13) should be solved.

4. Inverse problem solution

As it was mentioned above, the parameters appearing in the mathematical model of casting solidification are known except the latent heats Q_{aus} and Q_{eu}. It is assumed that the temperature values T_{dl}^f at the points $x^1 = x^{i*}$, $x^2 = x^{j*}$ (c.f. equation (9)) located in the casting sub-domain for times t^f are known

$$
T_{dl}^f = T_d\left(x^l, t^f\right), \quad l = 1, 2, \quad f = 1, 2, \ldots, F \tag{15}
$$

To solve the inverse problem the least squares criterion is applied

$$
S\left(Q_{aus}, Q_{eu}\right) = \frac{1}{2F}\sum_{l=1}^{2}\sum_{f=1}^{F}\left(T_l^f - T_{dl}^f\right)^2 \tag{16}
$$

where T_{dl}^f and $T_l^f = T(x^l, {}^f)$ are the measured and estimated temperatures, respectively. The estimated temperatures are obtained from the solution of the direct problem (c.f. chapter 1) by using the current available estimate of these parameters e.g. from preliminary experiments.

In the case of typical gradient method application [2, 5, 6] the criterion (16) is differentiated with respect to the unknown parameters Q_{aus}, Q_{eu} and next the necessary condition of optimum is used.

Finally one obtains the following system of equations

$$\begin{cases} \dfrac{\partial S}{\partial Q_{aus}} = \dfrac{1}{F}\sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left(T_l^f - T_{dl}^f\right)\left(Z_{l1}^f\right)^k = 0 \\ \dfrac{\partial S}{\partial Q_{eu}} = \dfrac{1}{F}\sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left(T_l^f - T_{dl}^f\right)\left(Z_{l2}^f\right)^k = 0 \end{cases} \tag{17}$$

where k is the number of iteration.

Function T_l^f is expanded in a Taylor series about known values of $Q_{aus}{}^k$, $Q_{eu}{}^k$, this means

$$T_l^f = \left(T_l^f\right)^k + \left(Z_{l1}^f\right)^k\left(Q_{aus}^{k+1} - Q_{aus}^k\right) + \left(Z_{l2}^f\right)^k\left(Q_{eu}^{k+1} - Q_{eu}^k\right) \tag{18}$$

Putting (18) into (17) one obtains

$$\begin{cases} \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left[\left(Z_{l1}^f\right)^k\right]^2 Q_{aus}^{k+1} + \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left(Z_{l1}^f\right)^k\left(Z_{l2}^f\right)^k Q_{eu}^{k+1} = \\ \quad \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left[\left(Z_{l1}^f\right)^k\right]^2 Q_{aus}^{k} + \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left(Z_{l1}^f\right)^k\left(Z_{l2}^f\right)^k Q_{eu}^{k} + \\ \quad \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left[T_{dl}^f - \left(T_l^f\right)^k\right]\left(Z_{l1}^f\right)^k \\ \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left(Z_{l1}^f\right)^k\left(Z_{l2}^f\right)^k Q_{aus}^{k+1} + \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left[\left(Z_{l2}^f\right)^k\right]^2 Q_{eu}^{k+1} = \\ \quad \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left(Z_{l1}^f\right)^k\left(Z_{l2}^f\right)^k Q_{aus}^{k} + \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left[\left(Z_{l2}^f\right)^k\right]^2 Q_{eu}^{k} + \\ \quad \sum\limits_{l=1}^{2}\sum\limits_{f=1}^{F}\left[T_{dl}^f - \left(T_l^f\right)^k\right]\left(Z_{l2}^f\right)^k \end{cases} \tag{19}$$

This system of equations allows to find the values of $Q_{aus}{}^{k+1}$ and $Q_{eu}{}^{k+1}$. The iteration process is stopped when the assumed number of iterations K is achieved.

5. Example of computations

The casting-mould system shown in Figure 1 has been considered. The basic problem and additional problems connected with the sensitivity functions have been solved using the explicit scheme of FDM [1]. The regular mesh created by 25×15 nodes with constant step $h = 0.002$ m has been introduced, time step $\Delta t = 0.1$ s. The following input data have been assumed: $\lambda = 30$ W/(mK), $\lambda_m = 1$ W/(mK),

$c_L = 5.88$ MJ/(m^3 K), $c_S = 5.4$ MJ/(m^{3}K), c_m = 1.75 MJ/(m^3 K), pouring temperature T_0 = 1300°C, liquidus temperature T_L = 1250°C, border temperature T_E = 1160°C, solidus temperature T_S = 1110°C, initial mould temperature T_{m0} = 20°C.

The problem of optimal sensors location has been solved under the assumption that $Q_{aus}^{\ 0}$ = 900 MJ/m^3, $Q_{eu}^{\ 0}$ = 1000 MJ/m^3. The application of optimization procedure shows that the best sensors position corresponds to the nodes from casting domain marked by A and B in Figure 1.

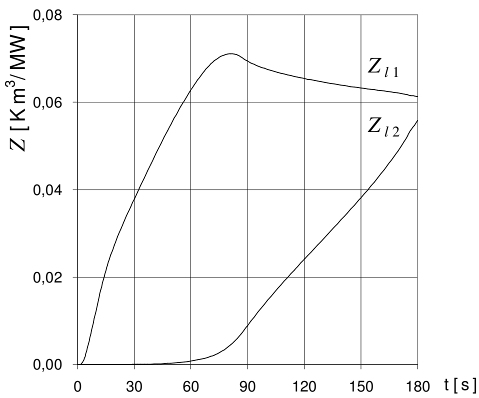

Fig. 2. Sensitivity functions (point A)

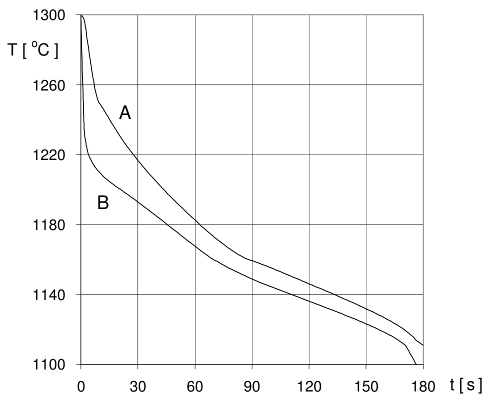

Fig. 3. Cooling curves

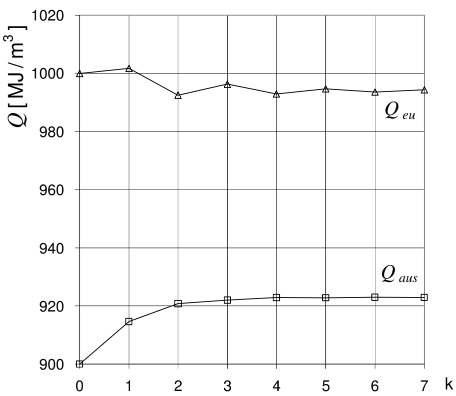

Fig. 4. Inverse problem solution

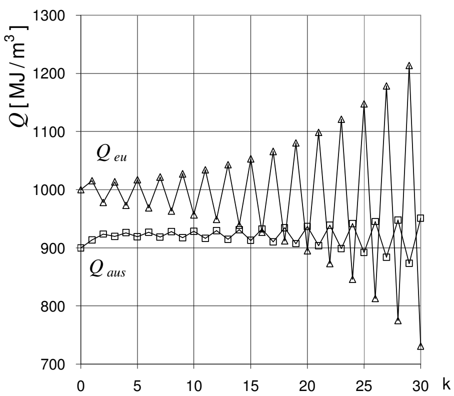

Fig. 5. Solution using points C and D

In Figure 2 the sensitivity functions at the point A for $Q_{aus}^{\ 0}$ and $Q_{eu}^{\ 0}$ are shown. Figure 3 illustrates the cooling curves at the points A, B obtained for the real values of Q_{aus}, Q_{eu}, this means $Q_{aus}^{\ 0}$ = 923 MJ/m^3, $Q_{eu}^{\ 0}$ = 994 MJ/m^3. Using these curves the inverse problem has been solved.

The successive iterations of $Q_{aus}^{\ k}$, $Q_{eu}^{\ k}$ are shown in Figure 4. It is visible that the iteration process is quickly convergent and the identified latent heats correspond to the previously assumed values.

The proper choice of sensors location seems to be very essential because it assures the effective and exact solution of identification problem. The good confirmation of this fact is the situation shown in Figure 5. In this Figure the successive steps of iteration process (equations (19)) are marked. One can see that the solution is not convergent and the identification of unknown parameters is impossible. The results shown in Figure 5 have been obtained for two randomly selected points (sensors) from casting domain (C and D in Figure 1). For the others positions of sensors location one can obtain the good values of searched parameters but the number of iterations will be probably greater than for optimal sensors position.

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References

[1] Mochnacki B., Suchy J.S., Numerical methods in computations of foundry processes, PFTA, Cracow 1995.

[2] Majchrzak E., Mendakiewicz J., Identification of cast iron substitute thermal capacity, Archives of Foundry 2006, 6, 22, 310-315.

[3] Patan M., Uciński D., Optimal location of sensors for parameter estimation of static distributed systems, R.Wyrzykowski et al. (eds.), PPAM 2001, LNCS 2328, 2002, 729-737.

[4] Dowding K.J., Blackwell B.F., Design experiments to estimate temperature dependent thermal properties, Inverse Problems in Engineering: Theory and Practice 1999, ASME 509-518.

[5] Majchrzak E., Mochnacki B., Identification of thermal properties of the system casting - mould, Mat. Science Forum 2007, 539-543, 2491-2496.

[6] Mochnacki B., Majchrzak E., Szopa R., Suchy J.S., Inverse problems in the thermal theory of foundry, Scientific Research of the Institute of Mathematics and Computer Science 2006, 1(5), 154-179.

[7] Kleiber M., Parameter sensitivity, J.Wiley & Sons Ltd., Chichester 1997.

[8] Mendakiewicz J., Application of sensitivity analysis in experiments design, Scientific Research of the Institute of Mathematics and Computer Science, Czestochowa University of Technology 2008, 1(7), 141-148.