EXPERIMENT DESIGN FOR PARAMETERS ESTIMATION OF NONLINEAR POISSON EQUATION - PART II

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Abstract. In this part of the paper the inverse problem consisting in the identification of unknown parameters $p_1, p_2$ appearing in conductivity $D(x) = p_1 x_1 x_2 + p_2$ where $x = \{x_1, x_2\}, -1 \leq x_1, x_2 \leq 1$ is analyzed. To solve this problem the additional information connected with the knowledge of function $U(x)$ at the set of points (sensors) selected from the domain considered is necessary. The fundamental problem is the selection of sensors location and here the algorithm assuring the optimal sensors location is presented. In the final part of the paper the results of computations are shown.

1. Formulation of the problem

The following two-dimensional Poisson equation is considered [1]

$$\begin{align*}
(x_1, x_2) &\in \Omega: \frac{\partial}{\partial x_1} \left[ (p_1 x_1 x_2 + p_2) \frac{\partial U(x_1, x_2)}{\partial x_1} \right] + \\
&+ \frac{\partial}{\partial x_2} \left[ (p_1 x_1 x_2 + p_2) \frac{\partial U(x_1, x_2)}{\partial x_2} \right] + Q(x_1, x_2) = 0
\end{align*}$$

(1)

where $\Omega = \{x_1, x_2: -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$, $Q(x_1, x_2)$ is the source function and $p_1, p_2$ are the unknown parameters.

The equation (1) is supplemented by boundary condition

$$(x_1, x_2) \in \Gamma: \quad U(x_1, x_2) = 0$$

(2)

To identify the parameters $p_1, p_2$ the additional information connected with the knowledge of function $U(x)$ at the set of points (sensors) selected from the domain considered is necessary. The accuracy of identification depends significantly on the choice of sensors location and this problem is here discussed.
2. Algorithm of optimal sensors location

Let \( X = \{ x^1, x^2, \ldots, x^M \} \) denotes the set of spatial points at which measurements may be taken. The practical design problem consists in selection of corresponding weights \( w_1, w_2, \ldots, w_M \) which define the best experimental conditions [1]. To solve this problem the following iterative algorithm under the assumption that number of unknown parameters equals 2 and number of sensors equals \( N \) can be applied.

At first, the sensitivity matrix is constructed

\[
Z = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22} \\
\vdots & \vdots \\
Z_{M1} & Z_{M2}
\end{bmatrix}
\]

where

\[
Z_{ij} = \frac{\partial U(x^l, p_1^0, p_2^0)}{\partial p_i^0}, \quad Z_{12} = \frac{\partial U(x^l, p_1^0, p_2^0)}{\partial p_2^0}, \quad l = 1, 2, \ldots, M
\]

and \( p_1^0, p_2^0 \) are the estimates of unknown parameters available e.g. from preliminary experiments.

Let

\[
Z(x') = [Z_{11} Z_{12}]
\]

and

\[
S(x') = Z^T(x') Z(x') = \begin{bmatrix}
Z_{11} Z_{12} \\
Z_{12} Z_{22}
\end{bmatrix}
\]

Step 1. Let \( k = 0 \) and \( \text{eps} \) is some positive tolerance. We assume that \( S_0 = \{ x^1, x^2, \ldots, x^N \} \) denotes the initial sub-set of \( X \).

Step 2. We set

\[
w^k_l = \begin{cases}
\frac{1}{M} & \text{if } x' \in S_k, \quad l = 1, 2, \ldots, M \\
0 & \text{if } x' \not\in X \setminus S_k
\end{cases}
\]

Step 3. The following matrices are calculated

\[
R(w^k) = Z^T w^k Z
\]
this means

$$\mathbf{R}(\mathbf{w}^k) = \begin{bmatrix} \sum_{i=1}^{M} w_{i}^{k} Z_{i1}^2 & \sum_{i=1}^{M} w_{i}^{k} Z_{i1} Z_{i2} \\ \sum_{i=1}^{M} w_{i}^{k} Z_{i1} Z_{i2} & \sum_{i=1}^{M} w_{i}^{k} Z_{i2}^2 \end{bmatrix}$$ \tag{9}$$

and

$$\mathbf{P}(x^i, \mathbf{w}^k) = \mathbf{R}^{-1}(\mathbf{w}^k) \mathbf{S}(x^i)$$ \tag{10}$$

where

$$\mathbf{w}^k = \begin{bmatrix} w_{1}^{k} & 0 & \ldots & 0 \\ 0 & w_{2}^{k} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & w_{M}^{k} \end{bmatrix}$$ \tag{11}$$

**Step 4.** Find

$$x^{*k} = \min_{x^i \in S_k} \left[ \text{trace} \mathbf{P}(x^i, \mathbf{w}^k) \right]$$ \tag{12}$$

and

$$x^{**k} = \max_{x^i \in X\backslash S_k} \left[ \text{trace} \mathbf{P}(x^i, \mathbf{w}^k) \right]$$ \tag{13}$$

**Step 5.** If

$$\text{trace} \mathbf{P}(x^{**k}, \mathbf{w}^k) - \text{trace} \mathbf{P}(x^{*k}, \mathbf{w}^k) > \text{eps}$$ \tag{14}$$

then set

$$S_{k+1} = \{ S_k \backslash \{x^{*k}\} \} \cup \{x^{**k}\}$$ \tag{15}$$

increase $k$ by one and go to **Step 2**, otherwise **Stop**.

It should be pointed out that the sensitivity coefficients (4) are determined using the direct differentiation method presented in the first part of the paper [2].

### 3. Results of computations

The set of possible sensors location is determined by the following points
The initial set of measurement points (20 sensors) is described as

\[ X = \left\{ x : x_{i,j} = -1 + \frac{2}{9} (i - 1), x_{2,j} = -1 + \frac{2}{9} (j - 1), \quad i, j = 1, 2, ..., 10 \right\} \quad (16) \]

The initial set of measurement points (20 sensors) is described as

\[ S_0 = \left\{ x : x_i = -1 + \frac{4}{9} (i - 1), \quad i = 1, 2, ..., 5, \quad x_{2,j} = -1 + \frac{6}{9} (j - 1), \quad j = 1, 2, 3, 4 \right\} \quad (17) \]

In Figure 1 the initial location of sensors is marked by black circles, while Figure 2 illustrates the optimal sensors location obtained using the algorithm above presented.

![Fig. 1. Initial position of sensors](image1)

![Fig. 2. Optimum position of sensors](image2)

**References**
