EXPERIMENT DESIGN FOR PARAMETERS ESTIMATION OF NONLINEAR POISSON EQUATION - PART I

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Abstract. In this part of the paper the nonlinear Poisson equation is considered, this means the conductivity is a function of the form $D(x) = p_1 x_1 x_2 + p_2$, where p_1, p_2 are the parameters and $x = \{x_1, x_2\}$, $-1 \le x_1,\ x_2 \le 1$ are the spatial co-ordinates. Sensitivity analysis with respect to parameters p_1, p_2 using the direct differentiation approach is discussed. The basic problem and additional ones are solved using the finite difference method. In the final part of the paper the results of computations are shown.

1. Formulation of the problem

The two-dimensional elliptic equation is considered

$$x \in \Omega: \qquad \nabla\left[D(x)\,\nabla U(x)\right] + Q(x) = 0 \tag{1}$$

where $D(x)$ is the coefficient of conductivity, $Q(x)$ is the source function, $x =$ $= \{x_1, x_2\}$, $\Omega = \{x_1, x_2: -1 \le x_1 \le 1, -1 \le x_2 \le 1\}$. The function $Q(x)$ is defined as follows

$$Q(x) = 10\exp\left[(x_1 - 0.75)^2 + (x_2 - 0.75)^2\right] \tag{2}$$

while the conductivity coefficient is expressed as

$$D(x) = p_1\, x_1\, x_2 + p_2 \tag{3}$$

where p_1, p_2 are the parameters ($p_1 = 1.05$, $p_2 = 4.09$).

The equation (1) is supplemented by Dirichlet boundary condition

$$x \in \Gamma: \quad U(x) = 0 \tag{4}$$

It should be pointed out that the mathematical model presented above taken from [1] is connected with computer-assisted tomography.

The aim of investigations is to solve the problem formulated and to determine the sensitivity functions $\partial U/\partial p_1$, $\partial U/\partial p_2$ using the direct differentiation method.

2. Sensitivity models

The equation (1) in Cartesian co-ordinate takes the form

$$\frac{\partial}{\partial x_1}\left[(p_1x_1x_2+p_2)\frac{\partial U(x_1,x_2)}{\partial x_1}\right]+\frac{\partial}{\partial x_2}\left[(p_1x_1x_2+p_2)\frac{\partial U(x_1,x_2)}{\partial x_2}\right]+Q(x_1,x_2)=0 \quad (5)$$

or

$$\begin{gathered}(p_1x_1x_2+p_2)\left[\frac{\partial^2 U(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 U(x_1,x_2)}{\partial x_2^2}\right]+\\ p_1x_2\frac{\partial U(x_1,x_2)}{\partial x_1}+p_1x_1\frac{\partial U(x_1,x_2)}{\partial x_2}+Q(x_1,x_2)=0\end{gathered} \quad (6)$$

At first, the equation (6) is differentiated with respect to the parameter p_1 and then

$$\begin{gathered}x_1x_2\left[\frac{\partial^2 U(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 U(x_1,x_2)}{\partial x_2^2}\right]+\\ (p_1x_1x_2+p_2)\left[\frac{\partial^2 V(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 V(x_1,x_2)}{\partial x_2^2}\right]+\\ x_2\frac{\partial U(x_1,x_2)}{\partial x_1}+p_1x_2\frac{\partial V(x_1,x_2)}{\partial x_1}+x_1\frac{\partial U(x_1,x_2)}{\partial x_2}+p_1x_1\frac{\partial V(x_1,x_2)}{\partial x_2}=0\end{gathered} \quad (7)$$

or

$$\begin{gathered}(p_1x_1x_2+p_2)\left[\frac{\partial^2 V(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 V(x_1,x_2)}{\partial x_2^2}\right]+\\ p_1x_2\frac{\partial V(x_1,x_2)}{\partial x_1}+p_1x_1\frac{\partial V(x_1,x_2)}{\partial x_2}+\\ x_1x_2\left[\frac{\partial^2 U(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 U(x_1,x_2)}{\partial x_2^2}\right]+x_2\frac{\partial U(x_1,x_2)}{\partial x_1}+x_1\frac{\partial U(x_1,x_2)}{\partial x_2}=0\end{gathered} \quad (8)$$

where $V(x_1,x_2)=\partial U(x_1,x_2)/\partial p_1$.

In similar way the equation (6) is differentiated with respect to p_2, namely

$$\frac{\partial^2 U(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 U(x_1,x_2)}{\partial x_2^2}+(p_1x_1x_2+p_2)\left[\frac{\partial^2 W(x_1,x_2)}{\partial x_1^2}+\frac{\partial^2 W(x_1,x_2)}{\partial x_2^2}\right]=0 \quad (9)$$

or

$$(p_1 x_1 x_2 + p_2)\left[\frac{\partial^2 W(x_1,x_2)}{\partial x_1^2} + \frac{\partial^2 W(x_1,x_2)}{\partial x_2^2}\right] + \frac{\partial^2 U(x_1,x_2)}{\partial x_1^2} + \frac{\partial^2 U(x_1,x_2)}{\partial x_2^2} = 0 \quad (10)$$

where $W(x_1,x_2) = \partial U(x_1,x_2)/\partial p_2$.

The sensitivity equations (8) and (10) are supplemented by Dirichlet conditions (c.f. equation (4)).

$$(x_1,x_2)\in\Gamma: \quad V(x_1,x_2)=0, \quad W(x_1,x_2)=0 \quad (11)$$

3. Method of solution and results of computations

The basic equation (6) and sensitivity ones (8), (10) are solved by means of the finite difference method. The spatial mesh created by the nodes

$$X = \left\{x:\ x_{1i} = -1 + \frac{2}{9}(i-1),\ x_{2j} = -1 + \frac{2}{9}(j-1),\ i, j = 1,2,...,10\right\} \quad (12)$$

is shown in Figure 1.

The approximation of equation (6) is the following

$$\begin{gathered}(p_1 x_{1i} x_{2j} + p_2)\left(\frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h^2} + \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h^2}\right) + \\ p_1 x_{2j}\frac{U_{i+1,j} - U_{i-1,j}}{2h} + p_1 x_{1i}\frac{U_{i,j+1} - U_{i,j-1}}{2h} + Q(x_{1i}, x_{2j}) = 0\end{gathered} \quad (13)$$

where h = 2/9 is the grid step.
From equation (13) results that

$$\begin{gathered}U_{ij} = \frac{1}{4}(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}) + \frac{h\, p_1\, x_{2j}}{8(p_1 x_{1i} x_{2j} + p_2)}(U_{i+1,j} - U_{i-1,j}) + \\ \frac{h\, p_1\, x_{1i}}{8(p_1 x_{1i} x_{2j} + p_2)}(U_{i,j+1} + U_{i,j-1}) + \frac{h^2}{4(p_1 x_{1i} x_{2j} + p_2)}Q(x_{1i}, x_{2j})\end{gathered} \quad (14)$$

The system of equations (14) can be solved using iteration process

$$
\begin{gathered}
U_{ij}^{k+1} = \frac{1}{4}\left(U_{i-1,\,j}^{k} + U_{i+1,\,j}^{k} + U_{i,\,j-1}^{k} + U_{i,\,j+1}^{k}\right) + \\
\frac{h\,p_1}{8(p_1 x_{1i} x_{2j} + p_2)}\left[x_{2j}\left(U_{i+1,\,j}^{k} - U_{i-1,\,j}^{k}\right) + x_{1i}\left(U_{i,\,j+1}^{k} - U_{i,\,j-1}^{k}\right)\right] + \\
\frac{h^2}{4(p_1 x_{1i} x_{2j} + p_2)} Q(x_{1i}, x_{2j})
\end{gathered}
\tag{15}
$$

where k is the number of iteration.

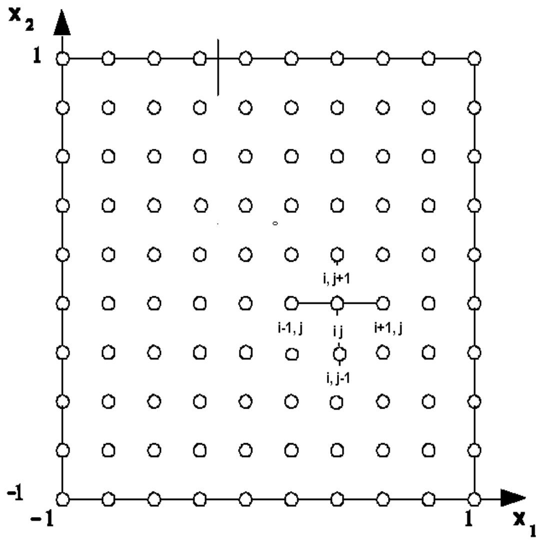

Fig. 1. The mesh

In similar way the approximation of equation (8) is constructed

$$
\begin{gathered}
(p_1 x_{1i} x_{2j} + p_2)\left(\frac{V_{i-1,\,j} - 2V_{ij} + V_{i+1,\,j}}{h^2} + \frac{V_{i,\,j-1} - 2V_{ij} + V_{i,\,j+1}}{h^2}\right) + \\
p_1 x_{2j}\frac{V_{i+1,\,j} - V_{i-1,\,j}}{2h} + p_1 x_{1i}\frac{V_{i,\,j+1} - V_{i,\,j-1}}{2h} + \\
x_{1i} x_{2j}\left(\frac{U_{i-1,\,j} - 2U_{ij} + U_{i+1,\,j}}{h^2} + \frac{U_{i,\,j-1} - 2U_{ij} + U_{i,\,j+1}}{h^2}\right) + \\
x_{2j}\frac{U_{i+1,\,j} - U_{i-1,\,j}}{2h} + x_{1i}\frac{U_{i,\,j+1} - U_{i,\,j-1}}{2h} = 0
\end{gathered}
\tag{16}
$$

and next

$$
\begin{aligned}
V_{i\,j}^{k+1} &= \frac{1}{4}\left(V_{i-1,\,j}^{k}+V_{i+1,\,j}^{k}+V_{i,\,j-1}^{k}+V_{i,\,j+1}^{k}\right)+\\
&\frac{h\,p_1}{8(p_1 x_{1i} x_{2\,j}+p_2)}\left[x_{2\,j}(V_{i+1,\,j}^{k}-V_{i-1,\,j}^{k})+x_{1\,i}(V_{i,\,j+1}^{k}-V_{i,\,j-1}^{k})\right]+\\
&\frac{x_{1i} x_{2\,j}}{4(p_1 x_{1i} x_{2\,j}+p_2)}\left(U_{i-1,\,j}^{k}+U_{i+1,\,j}^{k}+U_{i,\,j-1}^{k}+U_{i,\,j+1}^{k}-4U_{i,j}^{k}\right)+\\
&\frac{h}{8(p_1 x_{1i} x_{2\,j}+p_2)}\left[x_{2\,j}(U_{i+1,\,j}^{k}-U_{i-1,\,j}^{k})+x_{1\,i}(U_{i,\,j+1}^{k}-U_{i,\,j-1}^{k})\right]
\end{aligned}
\tag{17}
$$

The approximation of equation (10) has following form

$$
\begin{aligned}
&(p_1 x_{1i} x_{2\,j}+p_2)\left(\frac{W_{i-1,\,j}-2W_{i\,j}+W_{i+1,\,j}}{h^2}+\frac{W_{i,\,j-1}-2W_{i\,j}+W_{i,\,j+1}}{h^2}\right)+\\
&\frac{U_{i-1,\,j}-2U_{i\,j}+U_{i+1,\,j}}{h^2}+\frac{U_{i,\,j-1}-2U_{i\,j}+U_{i,\,j+1}}{h^2}=0
\end{aligned}
\tag{18}
$$

and then

$$
\begin{aligned}
W_{i\,j}^{k+1} &= \frac{1}{4}\left(W_{i-1,\,j}^{k}+W_{i+1,\,j}^{k}+W_{i,\,j-1}^{k}+W_{i,\,j+1}^{k}\right)+\\
&\frac{1}{4(p_1 x_{1i} x_{2\,j}+p_2)}\left(U_{i-1,\,j}^{k}+U_{i+1,\,j}^{k}+U_{i,\,j-1}^{k}+U_{i,\,j+1}^{k}-4U_{i\,j}^{k}\right)
\end{aligned}
\tag{19}
$$

In Figure 2 the distribution of function $U\,(x_1, x_2)$ is shown, while the Figures 3 and 4 illustrate the distribution of sensitivity functions $V\,(x_1, x_2)$ and $W\,(x_1, x_2)$.

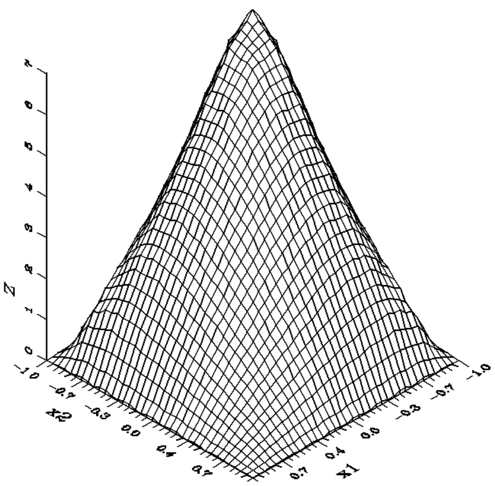

Fig. 2. Distribution of function U

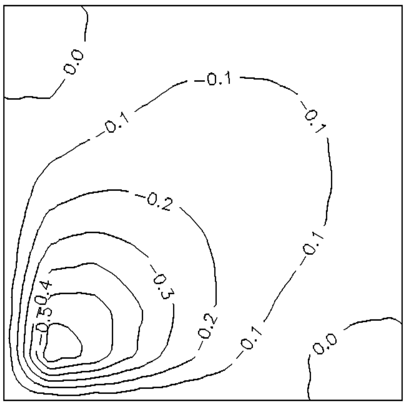

Fig. 3. Distribution of function V

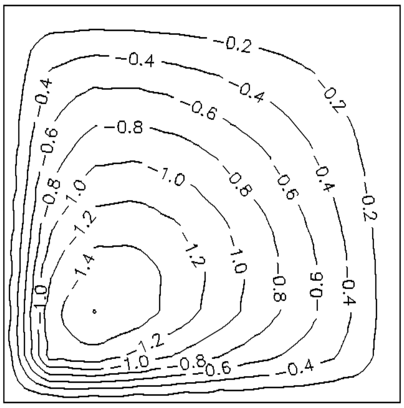

Fig. 4. Distribution of function W

References

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