CURVILINEAR FINITE DIFFERENCE METHOD (CFD) APPROXIMATION OF DIFFERENTIAL OPERATORS

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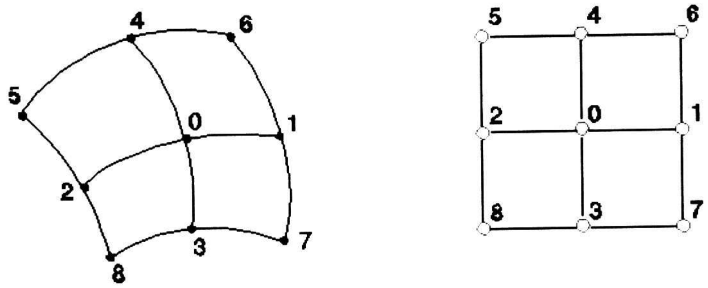
Abstract. Curvilinear finite difference method is a one of variants of generalized finite difference method. Geometrical mesh can be created by the optional set of points for which the *n*-points stars are defined. In this paper the 9-points stars are considered (2D task) and the method of differential operators approximation is presented. In the final part of the paper the example of computations is shown.

1. CFD for irregular 9-points stars

Let us consider 2D domain Ω (a real system $\{x, y\}$) covered by irregular mesh creating 9-points stars - as in Figure 1a. We assume that the successive stars can be univocally transformed on the regular rectangular (square) stars in the so-called ”mother system $\{\xi, \eta\}$” (see [1-3]) - as in Figure 1b.

The function for which the local approximation of first and second order derivatives is searched we denote by U. The nodes creating the 9-points star (a local numeration) are denoted by 0 (central node) and 1, 2,...,8 (adjacent nodes).



real system mother system

Fig. 1. Real and mother stars

The function $\mathbf{U}(\xi,\eta)$ is approximated by the algebraic polynomial in the form

$$\mathbf{U}(\xi,\eta)=\mathbf{P}\cdot\mathbf{a} \tag{1}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^2\eta & \xi\eta^2 & \xi^2\eta^2 \end{bmatrix} \tag{2}$$

is the interpolation base, whereas

$$\mathbf{a} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_8 \end{bmatrix}^{\mathrm{T}} \tag{3}$$

is a vector of coefficients. Next the following system of equations is considered

$$\mathbf{A} \cdot \mathbf{a} = \mathbf{u} \tag{4}$$

where

$$\mathbf{u} = \begin{bmatrix} u_0 & u_1 & u_2 & \dots & u_8 \end{bmatrix}^{\mathrm{T}} \tag{5}$$

is a vector of function U values at the points corresponding to the star nodes, at the same time

$$\mathbf{A} = \begin{bmatrix} 1 & \xi_0 & \eta_0 & \dots & \xi_0^2 & \eta_0^2 \\ 1 & \xi_1 & \eta_1 & \dots & \xi_1^2 & \eta_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \xi_8 & \eta_8 & \dots & \xi_8^2 & \eta_8^2 \end{bmatrix} \tag{6}$$

is a main matrix of system (4) resulting from the natural interpolation conditions. This system of equations allows to determine the coefficients a_i and to find the approximation of $\mathbf{U}(\xi,\eta)$ in a form of polynomial. So, if det $\mathbf{A} \neq 0$, then

$$\mathbf{U}(\xi,\eta) = \mathbf{P} \cdot \mathbf{A}^{-1} \cdot \mathbf{u} = \mathbf{N} \cdot \mathbf{u} = F(\mathbf{u},\xi,\eta) \tag{7}$$

while **N** is a matrix of shape functions in a local co-ordinate system.
It should be pointed out that in a case of rectangular mesh (e.g. square mesh with step $h=1$), the form of matrix **A** is a very simple one.
The same result can be obtained using the Lagrange interpolation [4], namely

$$\mathbf{U}(\xi,\eta) = \left(\begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & \eta & \eta^2 \end{bmatrix}\right)\left(\mathbf{A}_1^{-1} \otimes \mathbf{A}_2^{-1}\right) \cdot \mathbf{u} = F(\mathbf{u},\xi,\eta) \tag{8}$$

where $\mathbf{A}_1^{-1}$, $\mathbf{A}_2^{-1}$ are the regular Lagrange matrices [4]. The symbol used in the last formula corresponds to the tensor product.

Representation of the mother star in a system { ξ,η } on the real star in a system {*x, y*} can be realized by the same transformation *F*, as in a case of function *U* (the similar approach is applied for the finite element method). So

$$x(\xi,\eta)=\mathbf{N}\cdot\mathbf{x}, \qquad y(\xi,\eta)=\mathbf{N}\cdot\mathbf{y} \tag{9}$$

where

$$\begin{aligned}\mathbf{x}&=\begin{bmatrix}x_0 & x_1 & x_2 & \dots & x_8\end{bmatrix}^{\mathrm{T}}\\ \mathbf{y}&=\begin{bmatrix}y_0 & y_1 & y_2 & \dots & y_8\end{bmatrix}^{\mathrm{T}}\end{aligned} \tag{10}$$

Differentiation of formulas (7) or (8) allows to determine the values of derivatives at the central point of mother star. Let us denote by $\partial\mathbf{U}$ the vector of partial derivatives of *U* with respect to ξ and η, this means

$$\partial\mathbf{U}=\begin{bmatrix}\dfrac{\partial U}{\partial\xi} & \dfrac{\partial U}{\partial\eta} & \dfrac{\partial^2 U}{\partial\xi^2} & \dfrac{\partial^2 U}{\partial\eta^2} & \dfrac{\partial^2 U}{\partial\xi\partial\eta}\end{bmatrix}^{\mathrm{T}} \tag{11}$$

We obtain

$$\partial\mathbf{U}=\mathbf{H}\cdot\mathbf{A}^{-1}\cdot\mathbf{u} \tag{12}$$

where **H** is the matrix resulting from differentiation of the base (2) with respect to ξ, η, ξ^2, η^2, $\xi\eta$

$$\mathbf{H}=\begin{bmatrix}0 & 1 & 0 & 2\xi & \eta & 0 & 2\xi\eta & \eta^2 & 2\xi\eta^2\\ 0 & 0 & 1 & 0 & \xi & 2\eta & \xi^2 & 2\xi\eta & 2\xi^2\eta\\ 0 & 0 & 0 & 2 & 0 & 0 & 2\eta & 0 & 2\eta^2\\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2\xi & 2\xi^2\\ 0 & 0 & 0 & 0 & 1 & 0 & 2\xi & 2\eta & 4\xi\eta\end{bmatrix} \tag{13}$$

Now, the transformation of partial derivatives of *U* in the mother co-ordinates on the partial derivatives in the real co-ordinate system will be presented. In this place the several approaches can be applied. Here the rules of differentiation will be used and then

$$U=U\left[\xi(x,y),\eta(x,y)\right] \tag{14}$$

next

$$\frac{\partial U}{\partial x}=\frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial U}{\partial y}=\frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2}&=\frac{\partial^2 U}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2+2\frac{\partial^2 U}{\partial \xi\partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial^2 U}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2+\\&+\frac{\partial U}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial U}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}\end{aligned}\tag{15}$$

$$\begin{aligned}\frac{\partial^2 U}{\partial y^2}&=\frac{\partial^2 U}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2+2\frac{\partial^2 U}{\partial \xi\partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}+\frac{\partial^2 U}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2+\\&+\frac{\partial U}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}+\frac{\partial U}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 U}{\partial x\partial y}&=\frac{\partial^2 U}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+\frac{\partial^2 U}{\partial \xi\partial \eta}\left(\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)+\frac{\partial U}{\partial \xi}\frac{\partial^2 \xi}{\partial x\partial y}+\\&+\frac{\partial^2 U}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial U}{\partial \eta}\frac{\partial^2 \eta}{\partial x\partial y}\end{aligned}$$

or

$$\begin{bmatrix}U_x\\U_y\\U_{xx}\\U_{yy}\\U_{xy}\end{bmatrix}=\begin{bmatrix}\xi_x & \eta_x & 0 & 0 & 0\\ \xi_y & \eta_y & 0 & 0 & 0\\ \xi_{xx} & \eta_{xx} & \xi_x^2 & \eta_x^2 & 2\xi_x\eta_x\\ \xi_{yy} & \eta_{yy} & \xi_y^2 & \eta_y^2 & 2\xi_y\eta_y\\ \xi_{xy} & \eta_{xy} & \xi_x\xi_y & \eta_x\eta_y & \left(\eta_y\xi_x+\xi_y\eta_x\right)\end{bmatrix}\cdot\begin{bmatrix}U_\xi\\U_\eta\\U_{\xi\xi}\\U_{\eta\eta}\\U_{\xi\eta}\end{bmatrix}\tag{16}$$

Denoting by **G** the main matrix of above system we have

$$\mathrm{d}\mathbf{U}=\mathbf{G}\cdot\partial\mathbf{U}=\mathbf{G}\cdot\mathbf{H}\cdot\mathbf{A}^{-1}\cdot\mathbf{u}\tag{17}$$

In the last formula symbol d**U** corresponds to the vector of derivatives in the real system.
Now, the elements of matrix **G** should be calculated. The knowledge of inverse transformation

$$U = U\left[x(\xi,\eta),\ y(\xi,\eta)\right] \tag{18}$$

allows, using the basic rules of differentiation, to find the derivatives being the elements of matrix **G**.

2. Example of computations

The considerations presented in the chapter 1 can be a base for solution of the following task. In the real system the 9-points star is distinguished. The star nodes correspond to points **0** (1, 2), **1** (3, 3), **2** (−1, 1), **3** (4, 1), **4** (−2, 3), **5** (2, 0), **6** (6, 2), **7** (−4, 2), **8** (0, 4) - Figure 2a.

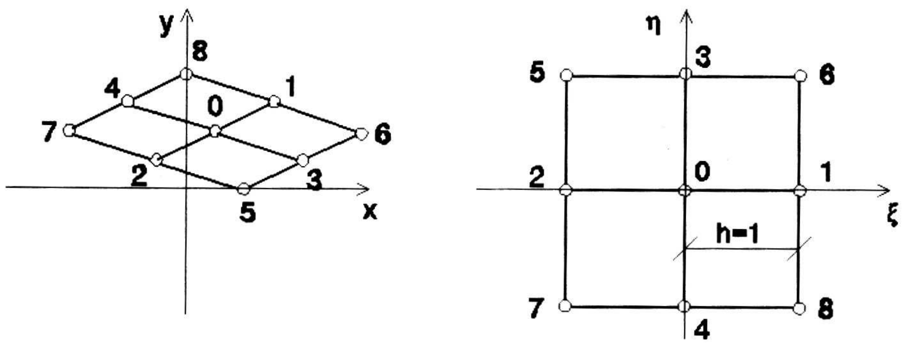


Fig. 2. Primary and transformed stars

It is easy to check that the formulas

$$\xi = \frac{x+3y-7}{5}$$
$$\eta = \frac{x-2y+3}{5} \tag{19}$$

transform the real star to unitary square one. The inverse transformation is of the form

$$\begin{cases} x = 1+2\xi+3\eta \\ y = 2+\xi-\eta \end{cases} \tag{20}$$

At first, the matrix **A** must be constructed. For the unitary square star it is the matrix

$$\mathbf{A}=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\1&1&0&1&0&0&0&0&0\\1&-1&0&1&0&0&0&0&0\\1&0&1&0&0&1&0&0&0\\1&0&-1&0&0&1&0&0&0\\1&-1&1&1&-1&1&-1&1&1\\1&1&1&1&1&1&1&1&1\\1&-1&-1&1&1&1&-1&-1&1\\1&1&-1&1&-1&1&1&-1&1\end{bmatrix} \tag{21}$$

The successive rows of matrix **A** correspond to values of basic functions at the nodes of unitary square star. We find the inverse matrix, namely

$$\mathbf{A}^{-1}=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&0.5&-0.5&0&0&0&0&0&0\\0&0&0&0.5&-0.5&0&0&0&0\\-1&0.5&0.5&0&0&0&0&0&0\\0&0&0&0&0&0.25&0.25&0.25&-0.25\\-1&0&0&0.5&0.5&0&0&0&0\\0&-0.5&0.5&0&0&-0.25&0.25&-0.25&0.25\\0&0&0&-0.5&0.5&0.25&0.25&-0.25&-0.25\\1&-0.5&-0.5&-0.5&-0.5&0.25&0.25&0.25&0.25\end{bmatrix} \tag{22}$$

We also determine the matrix **G**. For the data assumed it is the following matrix

$$\mathbf{G}=\begin{bmatrix}0.2&0.2&0&0&0\\0.6&-0.4&0&0&0\\0&0&0.04&0.04&0.08\\0&0&0.36&0.16&-0.48\\0&0&0.12&-0.08&0.04\end{bmatrix} \tag{23}$$

Application of formula (17) allows to calculate the values of derivatives at the nodes 0, 1,..., 8 of real star. To check the exactness of finite difference approximation it was assumed that the values of U at the nodes of real star are equal to

$$\mathbf{u} = \begin{bmatrix} 10 & 12 & 12 & 20 & 0 & 22 & 22 & 2 & 2 \end{bmatrix}^{\mathrm{T}} \tag{24}$$

Equation (17) for central node 0 gives the following values of derivatives [5]

$$\mathrm{d}\mathbf{U} = \begin{bmatrix} 2 & -4 & 0.16 & 1.44 & 0.48 \end{bmatrix}^{\mathrm{T}} \tag{25}$$

The second part of numerical experiment consisted in the direct polynomial interpolation of function U at the points oriented in the real system. The differentiation of this polynomial with respect to x and y gave for central node exactly the same values of partial derivatives.

Application of CFD for numerical solution of boundary problems requires the transformation of successive real stars to the mother system in which one can find the approximation of derivatives and the approximation of e.g. Laplace operator, next the obtained formulas are transformed to the real system. At present the authors are going to prepare the computer program basing on CFD for numerical solution of eliptic and parabolic equations. The others versions of generalized finite difference method are presented, among others, in [6, 7].

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