METHOD OF FUNDAMENTAL SOLUTIONS FOR HELMHOLTZ EIGENVALUE PROBLEMS IN ELLIPTICAL DOMAINS

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Abstract. In this paper, the method of fundamental solutions for Helmholtz eigenproblems in an elliptical domain is presented. To find the approximate solution of the problem, the Hankel function of the first kind and zero order as the fundamental solution of the Helmholtz equation in unbounded domain on the plane was used. Numerical examples illustrating the accuracy of the present method are given.

Introduction

The transverse vibration of a membrane which occupy the domain *S*, is governed by the following equation [1]

$$\nabla^2 w = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{1}$$

where ∇^2 is the Laplace operator, $c=\sqrt{T/\rho}$ is the speed of sound, T is the uniform tension per unit length of the edge ∂S, ρ is the density per unit area, $w(x,y,t)$ is the displacement of the plate point (x,y) at time t. When the free vibration of the membrane is considered, a harmonic time dependence of the form $e^{i\omega t}$ is assumed, i.e. $w(x,y,t)=W(x,y)e^{i\omega t}$. Substituting the form of the displacement into (1), the Helmholtz equation is obtained

$$\nabla^2 W + \Omega^2 W = 0, \quad (x,y)\in S \tag{2}$$

where $\Omega = \omega\sqrt{\rho/T}$. The equation (2) is completed by boundary conditions. We assume here the Dirichlet condition

$$W(x,y)=0, \quad (x,y)\in \partial S \tag{3}$$

The differential equation (2) and boundary condition (3) form the Helmholtz eigenvalue problem. The problem for elliptic domain S is the subject of this paper.

1. The method of fundamental solutions

We solve the eigenvalue problem (2)-(3) by using the method of fundamental solutions (MFS). The MFS is a boundary method which does not involve discretization and integration. The method is applicable to problems for which a fundamental solution of the governing equation is known. The basic idea of the method is the usage of a linear combination of fundamental solutions with sources located at fictitious points outside the domain of the considered problem.

The fundamental solution of a differential equation is not unique. For the Helmholtz equation (2) we take the Hankel function [2, 3]

$$G(P,Q)=\frac{i}{4}H_0^{(1)}(\Omega\, d(P,Q)) \tag{4}$$

where $H_0^{(1)}(\cdot)$ is the Hankel function of the first kind and zero order, $d(P,Q)$ is the distance between points P and Q, $i=\sqrt{-1}$. The functions $G(x,y;Q_k)$ satisfy the Helmholtz equation in the domain S for each source points Q_k located outside S. In the MFS, we approximate the solution of the problem by a function of the form

$$w_n(P)=\sum_{k=1}^{n} c_k\, G(P,Q_k) \tag{5}$$

The approximate solution w_n satisfies the differential equation (2), and it does not satisfy the boundary condition (3). The condition can be satisfied approximately by a suitable determination of the coefficients c_k, $k = 1, 2,\ldots,n$. For this purpose we use the last square method. First we choose the points P_i, $i = 1, 2,\ldots,n$, located on boundary ∂S of the domain S. We determine the coefficients c_k so that the function

$$f\left((c_1,c_2,\ldots,c_n)\right)=\sum_{i=1}^{n}\left[w_n(P_i;c_1,c_2,\ldots,c_n)\right]^2 \tag{6}$$

has a minimum. This leads to a homogeneous linear system of equations

$$\frac{\partial f}{\partial c_j}=2\sum_{k=1}^{n}c_k\sum_{i=1}^{n}G(P_i,Q_j)G(P_i,Q_k)=0,\quad j=1,2,\ldots,n \tag{7}$$

which can be written in the matrix form as

$$\mathbf{A}\,\mathbf{c}=\mathbf{0} \tag{8}$$

where $\mathbf{A}=\left[a_{jk}\right]_{1\le j,k\le n}$, $a_{jk}=\sum_{i=1}^{n}G(P_i,Q_j)G(P_i,Q_k)$, $c=\left[c_1\ \ c_2\ \ \ldots\ \ c_n\right]^T$

For a non-trivial solution of equation (8), the determinant of the coefficient matrix **A** is set equal to zero, yielding the eigenvalue equation

$$\det \mathbf{A}(\Omega) = 0 \tag{9}$$

Equation (9) with the unknown Ω, is solved numerically. The eigenfunctions for the determined eigenvalues Ω_m, m = 1, 2,…, are given by (5) where the coefficients c_k, k = 2,…,n, are derived in dependence of c_1 from n-1 equations of the system (7).

For the considered Helmholtz eigenvalue problem the function w_n defined by (5), after using (4) has the form

$$w_n(x, y) = \sum_{k=1}^{n} c_k H_0^{(1)}\left(\Omega\sqrt{(x-\xi_k)^2 + (y-\eta_k)^2}\right), \quad (x, y) \in S \tag{10}$$

where the source points $Q_k(\xi_k, \eta_k)$, k = 1, 2,…,n, are over the domain S. But the coefficients of the matrix **A** are

$$a_{jk} = \sum_{i=1}^{n} H_0^{(1)}\left(\Omega\sqrt{(x_i-\xi_j)^2 + (y_i-\eta_j)^2}\right) H_0^{(1)}\left(\Omega\sqrt{(x_i-\xi_k)^2 + (y_i-\eta_k)^2}\right) \tag{11}$$

where the collocation points $P(x_i, y_i)$, i = 1, 2,…,n, are located on the boundary ∂S of the domain S.

2. Numerical examples

We assume that the boundary of the considered membrane domain is an ellipse with major and minor semi-diameters equal a and b, respectively. The collocation points are uniformly distributed on this ellipse and the source points are uniformly distributed on a circle with the radius $r = \alpha \cdot \max\{a, b\}$, $\alpha > 1$. Geometry configuration of an elliptical domain with the collocation points on the boundary and a circle with the source points is presented in Figure 1.

The Hankel function $H_0^{(1)}$ occurring in equations (10, 11) assumes complex values and that way the elements of the matrix **A** are complex. Our goal is to evaluate real roots of the complex equation (9). For this purpose the new function F is defined as

$$F(\Omega) = |\det \mathbf{A}(\Omega)| \tag{12}$$

where the symbol $|\cdot|$ denotes a modulus of complex number. It is clear that the roots of equation (9) are determined by minima of the function F. In Figure 2 an exemplary graph of the function in logarithmic scale for an elliptical domain with semi-diameters ratio a/b = 2.0 and $n = 24$ is presented. Locations of the distinct drops indicate the roots of equation (9).

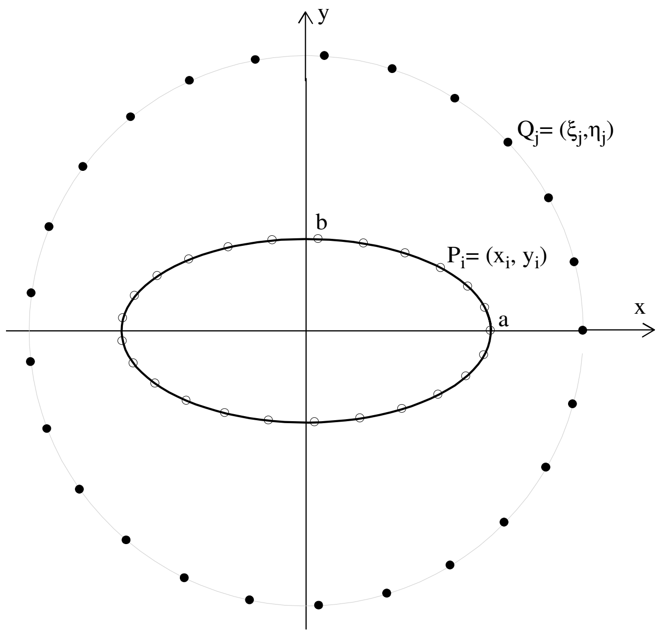

Fig. 1. Geometry configuration of the considered domain with collocation points P_i on the ellipse and source points on Q_j the circle

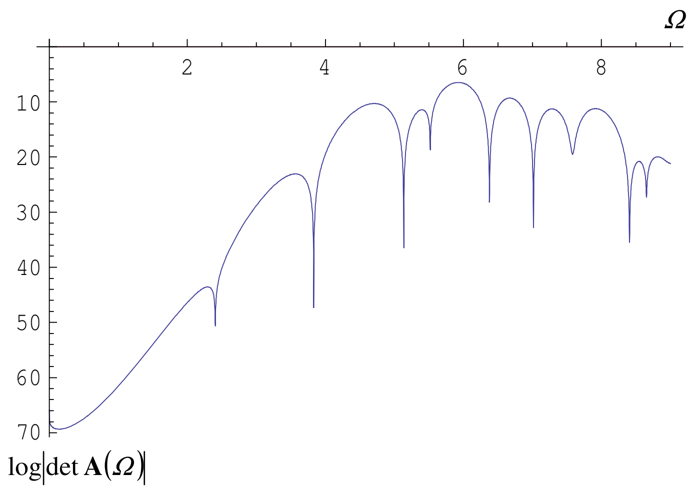

Fig. 2. Logarithmic values of modulus of the determinant det**A**(Ω) as a function of Ω for the Helmholtz operator in the elliptical domain with semi-diameters ratio a/b = 2.0

To estimate the MFS accuracy, an example of the Helmholtz eigenvalue problem in a circular domain will be solved. In this case the eigenvalues are the roots of the equation [3]

$$J_m(\Omega) = 0, \; m = 1, 2, \ldots \tag{13}$$

A sequence of the eigenvalues for each m can be obtained. First members of the sequences are numerically determined by finding the local minima of the function $log|F(\Omega)|$, where $F(\Omega)$ is given by equation (12). The calculations were performed for various numbers of source points ($n = 16$; 20; 24). The comparison of the results obtained by the MFS with exact values obtained as roots of equation (13) is presented in Table 1. Note that the numerical calculation by MFS with small number of the source points gives the results with a larger error and can lead to omission of eigenvalues. The comparison of the results obtained by MFS and the exact eigenvalues shown that the absolute error for $n = 24$ is not larger as 10^{-5}.

The eigenvalues of the Helmholtz operator are the frequency parameters of free vibration of a membrane. The first seven frequency parameter values of the elliptic membrane with clamped edge are presented in Table 2. The calculations are performed for various values of semi-diameters ratio a/b of the ellipse. The frequency parameters obtained by using MFS were compared with the eigenvalues determined by the finite element method in the paper [1] by Buchanan and Peddieson. For assumed number of sources $n = 24$ in MFS a small differences of the results calculated by using both methods are observed.

Table 1. Eigenvalues Ω_n ($n = 1,\ldots,10$) of the Helmholtz operator in a circular domain obtained by the MFS for $n = 16$; 20; 24 and exact eigenvalues obtained as roots of equation (12)

	MFS $n = 16$	MFS $n = 20$	MFS $n = 24$	Exact eigenvalues
Ω_1	2.40483	2.40483	2.40483	2.40483
Ω_2	3.83172	3.83171	3.83171	3.83171
Ω_3	5.13560	5.13562	5.13562	5.13562
Ω_4	5.52005	5.52008	5.52008	5.52008
Ω_5	6.37893	6.38014	6.38016	6.38016
Ω_6	7.01522	7.01558	7.01559	7.01559
Ω_7	7.58199	7.58809	7.58834	7.58834
Ω_8	8.40860	8.41726	8.41724	8.41724
Ω_9	8.65317	8.65374	8.65373	8.65373
Ω_{10}	–	8.76646	8.77148	8.77148

Table 2. Frequency parameters Ω_n (n = 1,…,7) of the **elliptic** membrane for various values of semi-diameters ratio b/a of the ellipse

	a/b = 1.5		a/b = 2.0		a/b = 3.0	
	MFS	FEM*	MFS	FEM*	MFS	FEM*
Ω_1	2.03911	2.039	1.88858	1.889	1.76311	1.763
Ω_2	2.91562	2.917	2.50508	2.508	2.14359	2.148
Ω_3	3.84097	3.850	3.42590	3.427	2.98058	3.072
Ω_4	4.78926	4.828	3.99047	4.000	3.42586	3.477
Ω_5	5.10279	5.104	4.58516	4.584	3.68419	3.697
Ω_6	5.74644	5.769	4.98843	4.990	4.04413	4.130
Ω_7	6.02787	6.127	5.54225	5.561	4.84154	4.821

the results by FEM are given in the paper [4]

Conclusions

In the paper, the application of the method of fundamental solutions to the Helmholtz eigenvalue problem in the elliptical domain has been presented. As the fundamental solution of the Helmholtz equation in the plane, the Hankel function of the first kind and zero order was used. In order to determine the eigenvalues, the minimum of a real function which is the logarithm of modulus of a complex function was found. The source points occurring in the approximate formula of the solution can be selected on a circle outside of the considered elliptical domain. The comparison of numerical results shows that high accuracy of the calculation is achieved for 24 sources.

References

[1] Pietrowski L., Równania różniczkowe cząstkowe, PWN, Warszawa 1955.

[2] Chen J.T., Chen I.L., Lee Y.T., Eigensolutions of multiply connected membranes using the method of fundamental solutions, Boundary Analysis with Boundary Elements 2005, 29, 166-174.

[3] Reutskiy S.Yu., The method of fundamental solutions for Helmholtz eigenvalue problems in simply and multiply connected domains, Boundary Analysis with Boundary Elements 2006, 30, 150-159.

[4] Buchanan G.R., Peddieson J., A finite element in eliptic coordinates with application to membrane vibration, Thin-Walled Structures 2005, 43, 1444-1454.

[5] Duffy D.G., Green's functions with applications, Chapman & Hall/CRC, Boca Raton, Washington DC 2001.