

## SOLUTION OF 2D HYPERBOLIC EQUATION BY MEANS OF THE BEM USING DISCRETIZATION IN TIME - ANALYSIS OF SOLUTION EXACTNESS

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**Abstract.** The hyperbolic equation (2D problem) supplemented by adequate boundary and initial conditions is considered. This equation is solved by means of the boundary element method using discretization in time. The aim of investigations is to analyze the influence of time step and the discretization assumed on the exactness of the obtained results.

### Introduction

The following problem is considered

$$\begin{aligned} \Omega = \{0 < x_1 < 1, 0 < x_2 < 1\} : & \quad \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} = \nabla^2 \theta \\ x_1 = 0, \quad 0 < x_2 < 1 : & \quad \theta = 1 \\ x_1 = 1, \quad 0 < x_2 < 1 : & \quad -\frac{\partial \theta}{\partial n} = 0 \\ x_2 = 0, \quad x_2 = 1, \quad 0 < x_1 < 1 : & \quad -\frac{\partial \theta}{\partial n} = 0 \\ t = 0 : & \quad \theta = 0 \\ t = 0 : & \quad \frac{\partial \theta}{\partial t} = 0 \end{aligned} \tag{1}$$

where  $\theta = \theta(x_1, x_2)$  is an unknown function,  $\{x_1, x_2\}$  are the spatial co-ordinates and  $t$  is the time.

The aim of investigations is to solve the problem by means of the boundary element method using discretization in time and to analyze the influence of time step and the discretization assumed on the results of computations.

## 1. Boundary element method using discretization in time

To solve the problem (1), the BEM using discretization in time is applied [1-4]. So, the time grid

$$0 = t^0 < t^1 < \dots < t^{f-2} < t^{f-1} < t^f < \dots < t^F < \infty \quad (2)$$

with constant step  $\Delta t = t^f - t^{f-1}$  is introduced.

The boundary integral equation corresponding to the problem (1) has the following form [1]

$$B(\xi)\theta(\xi, t^f) + \int_{\Gamma} \theta^*(\xi, x) q^f d\Gamma = \int_{\Gamma} q^*(\xi, x) \theta^f d\Gamma + \int_{\Omega} [B\theta^{f-1} - C\theta^{f-2}] \theta^*(\xi, x) d\Omega \quad (3)$$

where  $B(\xi) \in (0,1)$  is the coefficient connected with the position of the point  $\xi$  on the boundary  $\Gamma$ ,  $\theta^*(\xi, x)$  is the fundamental solution,  $q^*(\xi, x) = -\partial\theta^*(\xi, x)/\partial n$  and  $q^f = -\partial\theta^f/\partial n$ .

For 2D problem the functions  $\theta^*(\xi, x)$  and  $q^*(\xi, x)$  are of the form [5]

$$\theta^* = \frac{1}{2\pi} K_0(r\sqrt{A}) \quad (4)$$

and

$$q^* = \frac{d\sqrt{A}}{2\pi r} K_1(r\sqrt{A}) \quad (5)$$

where  $K_0(\cdot)$  and  $K_1(\cdot)$  are the modified Bessel's function of first kind, zero and first order, respectively,  $r$  is the distance between the points  $\xi$  and  $x$ , while

$$d = (x_1 - \xi_1)\cos\alpha_1 + (x_2 - \xi_2)\cos\alpha_2 \quad (6)$$

where  $\mathbf{n} = [\cos\alpha_1, \cos\alpha_2]$ . In equations (3), (4), (5):

$$A = \beta^2 + \beta, \quad B = 2\beta^2 + \beta, \quad C = \beta^2 \quad (7)$$

where  $\beta = 1/\Delta t$ .

To solve the boundary integral equation (3) the boundary  $\Gamma$  is divided into  $N$  constant elements and the interior  $\Omega$  is divided into  $L$  constant internal cells as shown in Figure 1.

	30	29	28	27	26	25	24	23	22	21	
31	131	132	133	134	135	136	137	138	139	140	20
32	121	122	123	124	125	126	127	128	129	130	19
33	111	112	113	114	115	116	117	118	119	120	18
34	101	102	103	104	105	106	107	108	109	110	17
35	91	92	93	94	95	96	97	98	99	100	16
36	81	82	83	84	85	86	87	88	89	90	15
37	71	72	73	74	75	76	77	78	79	80	14
38	61	62	63	64	65	66	67	68	69	70	13
39	51	52	53	54	55	56	57	58	59	60	12
40	41	42	43	44	45	46	47	48	49	50	11
	1	2	3	4	5	6	7	8	9	10	

Fig. 1. Discretization

At the first step of computations the following system of algebraic equation should be solved ( $i = 1, 2, \dots, N$ )

$$\sum_{j=1}^N G_{ij} q_j^f = \sum_{j=1}^N H_{ij} \theta_j^f + \sum_{l=1}^L P_{il} (B\theta_l^{f-1} - C\theta_l^{f-2}) \quad (8)$$

The definitions of elements  $G_{ij}$ ,  $H_{ij}$ ,  $P_{il}$  are presented in [2].

At the second step of computations the internal values of function  $\theta$  should be determined ( $i = N + 1, N + 2, \dots, N + L$ )

$$\theta_i^f = \sum_{j=1}^N H_{ij} \theta_j^f - \sum_{j=1}^N G_{ij} q_j^f + \sum_{l=1}^L P_{il} (B\theta_l^{f-1} - C\theta_l^{f-2}) \quad (9)$$

## 2. Analysis of solution exactness

Application of the boundary element method using discretization in time requires the proper assumption of time step  $\Delta t$  and number of internal cells  $L$ . The testing computations presented in this paper have been done for time step  $\Delta t = 0.05, 0.1, 0.5$  and number of internal cells  $L = 10 \times 10$  (c.f. Figure 1),  $20 \times 20, 40 \times 40$ . The successive variants of computations are collected in Table 1. To show convergence, the error of numerical solution is calculated

$$B^f = \frac{1}{L^2} \sum_{l=1}^L \left( \nabla_a^2 \theta_l^f - A \theta_l^f + B \theta_l^{f-1} + C \theta_l^{f-2} \right)^2 \quad (10)$$

where  $\nabla_a^2 \theta_l^f$  denotes the approximation of operator  $\nabla^2 \theta$  at the internal point  $l$ .

Table 1. Variants of computations

Variant	Mesh	Time step $\Delta t$	Error $E$
1	10×10	0.05	0.321
2	10×10	0.10	0.094
3	10×10	0.50	0.017
4	20×20	0.05	0.099
5	20×20	0.10	0.036
6	20×20	0.50	0.011
7	40×40	0.05	0.037
8	40×40	0.10	0.018
9	40×40	0.50	0.008

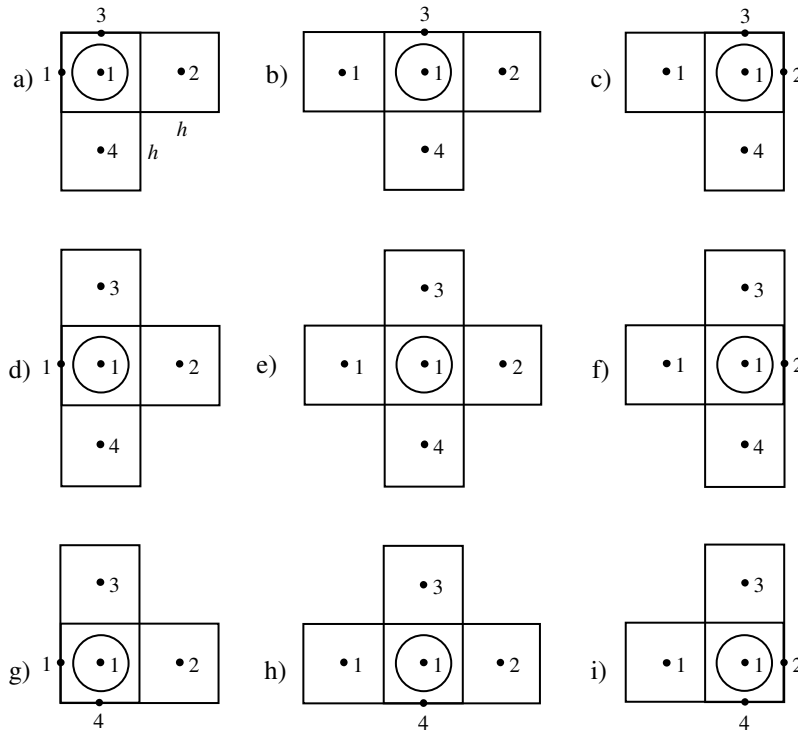
In Figure 2 different locations of internal point  $l$  are shown (c.f. Figure 1) and for these cases the following approximations of operator  $\nabla^2 \theta$  have been taken into account

$$\begin{aligned}
 \text{a) } \nabla_a^2 \theta_l^f &= \frac{8\theta_1^f - 12\theta_l^f + 4\theta_2^f}{3h^2} + \frac{8\theta_3^f - 12\theta_l^f + 4\theta_4^f}{3h^2} \\
 \text{b) } \nabla_a^2 \theta_l^f &= \frac{\theta_1^f - 2\theta_l^f + \theta_2^f}{h^2} + \frac{8\theta_3^f - 12\theta_l^f + 4\theta_4^f}{3h^2} \\
 \text{c) } \nabla_a^2 \theta_l^f &= \frac{4\theta_1^f - 12\theta_l^f + 8\theta_2^f}{3h^2} + \frac{8\theta_3^f - 12\theta_l^f + 4\theta_4^f}{3h^2} \\
 \text{d) } \nabla_a^2 \theta_l^f &= \frac{8\theta_1^f - 12\theta_l^f + 4\theta_2^f}{3h^2} + \frac{\theta_3^f - 2\theta_l^f + \theta_4^f}{h^2} \\
 \text{e) } \nabla_a^2 \theta_l^f &= \frac{\theta_1^f - 2\theta_l^f + \theta_2^f}{h^2} + \frac{\theta_3^f - 2\theta_l^f + \theta_4^f}{h^2} \\
 \text{f) } \nabla_a^2 \theta_l^f &= \frac{4\theta_1^f - 12\theta_l^f + 8\theta_2^f}{3h^2} + \frac{\theta_3^f - 2\theta_l^f + \theta_4^f}{3h^2}
 \end{aligned} \quad (11)$$

$$g) \nabla_a^2 \theta_l^f = \frac{8\theta_1^f - 12\theta_l^f + 4\theta_2^f}{3h^2} + \frac{4\theta_3^f - 12\theta_l^f + 8\theta_4^f}{3h^2}$$

$$h) \nabla_a^2 \theta_l^f = \frac{\theta_1^f - 2\theta_l^f + \theta_2^f}{h^2} + \frac{4\theta_3^f - 12\theta_l^f + 8\theta_4^f}{3h^2}$$

$$i) \nabla_a^2 \theta_l^f = \frac{4\theta_1^f - 12\theta_l^f + 8\theta_2^f}{3h^2} + \frac{4\theta_3^f - 12\theta_l^f + 8\theta_4^f}{3h^2}$$


 Fig. 2. Different locations of internal point  $l$ 

In Figures 3-5 the course of function  $\theta_l^f$  at the central point of domain considered for successive variants of computations is shown. Figure 6 illustrates the results for all variants. It is visible, that the results are strongly dependent on the time step assumed. In the last column of Table 1 the global error of numerical solution is presented, this means

$$E = \sqrt{\frac{1}{F} \sum_{f=1}^F B^f} \quad (12)$$

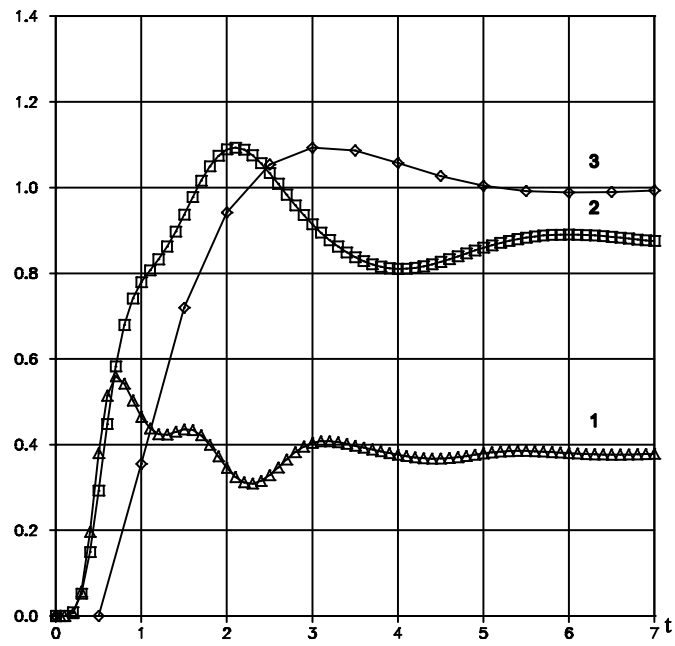


Fig. 3. Variants 1, 2, 3

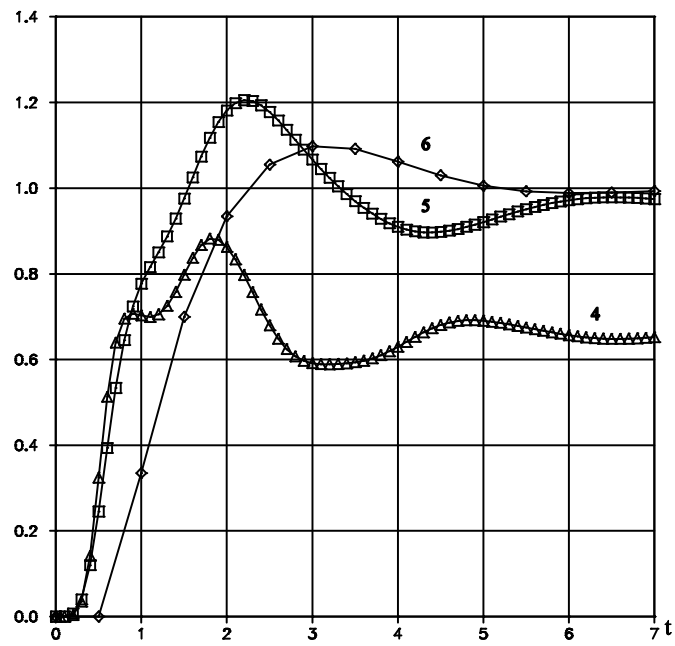


Fig. 4. Variants 4, 5, 6

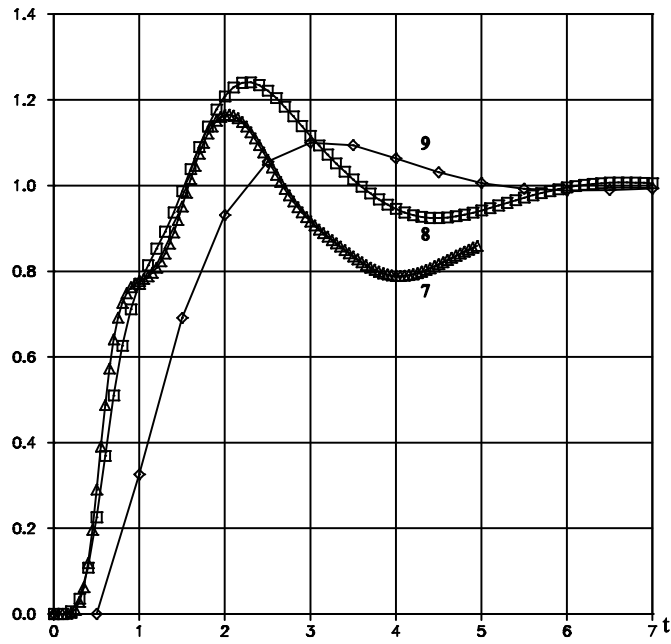


Fig. 5. Variants 7, 8, 9

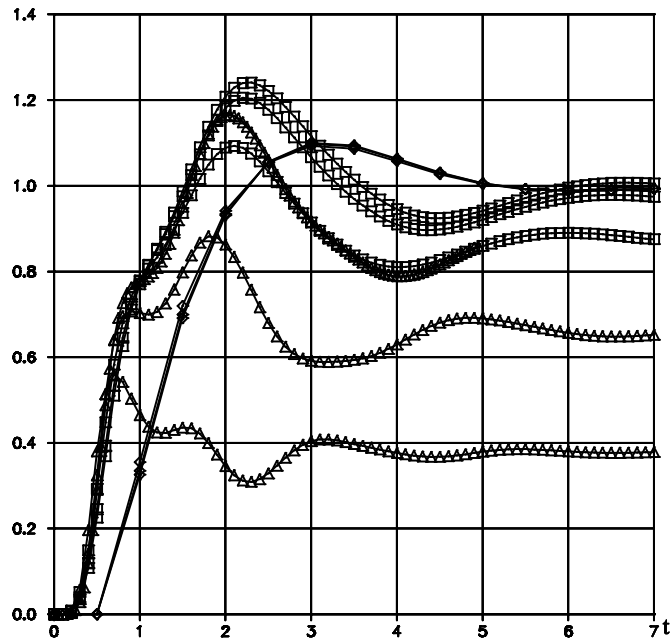


Fig. 6. All variants of computations

## Conclusions

For variants 3, 6 and 9 corresponding to the small values of error (12) the results of computations are practically the same.

It should be pointed out that the computations have been done for the time steps greater than the optimum time steps resulting from calculations (see: Table 1) and for these values the errors were very small.

Summing up, the BEM using discretization in time constitutes the effective numerical method of hyperbolic equation solution under the assumption that the time step is proper. Additionally, if the small value of time step is needed then the big number  $L$  of internal cells should be taken into account.

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