ON THE CERTAIN ASYMPTOTIC APPROACH TO THE MODELLING OF MICROHETEROGENEOUS MEDIA

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Abstract. The object of analysis is mathematical modelling of thermomechanical problems of microheterogeneous media. The aim of this contribution is to propose a certain asymptotic modeling technique. The proposed approach is realized both in consistent and semiconsistent form.

Introduction

The well known asymptotic homogenization theory leads to the solutions to what are called the periodic cell problems, [1]. Solution to these problems makes it possible to define the effective modulae of the medium under considerations. In most cases obtaining the solution to the cell problem is not an easy task and can not be realized in the analytical form. On the other hand the asymptotic modelling of differential equations introduced in this paper does not involve any periodic (or locally periodic) cell problem. However, the proposed asymptotic procedure introduces the extra unknown functions which are called fluctuation amplitudes. At the same time the consistent asymptotic modelling makes it possible to define effective modulae by eliminating these extra unknowns. Simultaneously, the semi-consistent asymptotic procedure describes the effect of the cell size on the overall behaviour of the medium under consideration in contrast to the asymptotic homogenization technique.

1. Basic concepts and notions

To make this paper self-consistent in this section we summarise the main concepts introduced in [2].

Let $\Omega\times\Xi$ be a bounded domain in R^n such that $\Omega\subset R^m$ and $\Xi\subset R^{n-m}$ provided that $n>m$. Points from domain Ω will be denoted by $x=(x_1,x_2,...,x_m)$ or $z=(z_1,z_2,...,z_m)$ and points from Ξ by $\xi=(\xi_1,\xi_2,...,\xi_{n-m})$. If $n=m$ then Ξ and ξ drop out from considerations. We introduce gradient operators of a smooth function f defined on $\Omega\times\Xi$ setting $\partial f\equiv grad_z f(z,\xi)$, $\overline{\nabla}\equiv grad_\xi f(z,\xi)$ for $n>m$ such that $\nabla f\equiv\partial f+\overline{\nabla}f$. When $n=m$ then $\overline{\nabla}$ drop out from considerations and the total gradient of f is denoted by $\nabla f=\partial f$. Moreover, we define the basic cell setting $\square\equiv[-\lambda_1/2,\lambda_1/2]\times...\times[-\lambda_m/2,\lambda_m/2]$ where $\lambda_1,...,\lambda_m>0$. By λ we denote diameter of $\square$, $\lambda\equiv diam\square$, and assume that $\lambda<<L_\Omega$, where L_Ω is the smallest characteristic length dimension of domain Ω. Hence $\square(x)\equiv x+\square$ is a cell with centre at $x\in R^m$. We also define $\Omega_x\equiv\Omega\cap\bigcup_{z\in\square(x)}\square(z)$, $x\in\overline{\Omega}$ as a cluster of 2^m cells having common sides. Family of cells $(\Omega,\square)=\{\square(x),x\in\overline{\Omega}\}$ will be referred to as the uniform cell distribution assigned to Ω.

In order to present an unified approach to the asymptotic modelling all subsequent considerations will be restricted to Sobolev spaces $H^k(\Omega)$ for $k=0,1$. By $H^0(\square)$ we denote the space of $\square$- periodic square-integrable functions defined in R^m. Moreover let $\tilde{f}^{(k)}(\cdot,\cdot)$ be a function defined in $\overline{\Omega}\times R^m$, $k=0,1,...,\alpha$. For the sake of simplicity the above denotations are assumed to hold both for scalar as well as vector functions.

Function $f\in H^1(\Omega)$ will be called the tolerance periodic function (with respect to cell $\square$ and tolerance parameter δ), $f\in TP^1_\delta(\Omega,\square)$, if for $k=0,1$ the following conditions hold

$$(i)\quad (\forall x\in\Omega)\left(\exists\tilde{f}^{(k)}(x,\cdot)\in H^0(\square)\right)\left[\left\|\partial^k f\big|_{\Omega_x}(\cdot)-\tilde{f}^{(k)}(x,\cdot)\right\|_{H^0(\Omega_x)}\le\delta\right],$$

$$(ii)\quad \int_{\square(\cdot)}\tilde{f}^{(k)}(\cdot,z)dz\in C^0(\overline{\Omega}).$$

Function $\tilde{f}^{(k)}(x,\cdot)$ will be referred to as the periodic approximation of $\partial^k f$ in $\square(x)$, $x\in\Omega$, $k=0,1$.

Function $h\in H^1(\Omega)$ will be called the highly oscillating function (with respect to the cell $\square$ and tolerance parameter δ), $h\in HO^1_\delta(\Omega,\square)$, if

$$(i)\quad h\in TP^1_\delta(\Omega,\square),$$

$$(ii)\quad (\forall x\in\Omega)\left[\tilde{h}^{(k)}(x,\cdot)\Big|_{\square(x)}=\partial^k\tilde{h}(x,\cdot),\ k=0,1\right].$$

Let $h=\left\{h^A(\cdot)\in HO_\delta^1(\Omega,\Box),\ A=1,...,N\right\}$ be a system of N linear independent functions which is assumed to be postulated *a priori* in every modelling problem under consideration. Functions $h^A(\cdot)$, $A=1,...,N$ are referred to as fluctuation shape functions. We shall assume that for every $x\in\overline{\Omega}$ condition $\left\langle\rho h^A\right\rangle(x)=0$ is satisfied for a certain given *a priori* positive function $\rho\in TP_\delta^0(\Omega,\Box)$. In a special case $\left\langle h^A\right\rangle(x)\equiv 0$, $A=1,...,N$.

The text of the subsequent sections is strictly related to that presented in the first part of monograph [3].

2. Consistent asymptotic averaging of integral functionals

Let us assume that $f(\cdot)\in HO_\delta^0(\Omega,\Box)$. We recall that if $f(\cdot)\in HO_\delta^0(\Omega,\Box)$ then for every $x\in\overline{\Omega}$ there exists function $\tilde{f}(x,z)$, $z\in\Box(x)$. Function $\tilde{f}(x,\cdot)$ is called a periodic approximation of highly oscillating function $f(\cdot)$ in $\Box(x)\cap\overline{\Omega}$. Firstly we introduce parameter $\varepsilon=1/n$, $n=1,2,...$. Moreover we define $\Omega_\varepsilon\equiv\left(-\varepsilon\frac{\lambda_1}{2},\varepsilon\frac{\lambda_1}{2}\right)\times...\times\left(-\varepsilon\frac{\lambda_m}{2},\varepsilon\frac{\lambda_m}{2}\right)$ as a scaled cell and $\Box_\varepsilon(x)\equiv x+\Box_\varepsilon$ as a scaled cell with a centre at $x\in\overline{\Omega}$. Let $\tilde{f}(x,\cdot)\in H^1(\Box)$ for every $x\in\overline{\Omega}$. We shall denote by $\tilde{f}_\varepsilon(x,\cdot)\in H^1(\Box_\varepsilon)\subset H^1(\Box)$ family of functions defined by

$$\tilde{f}_\varepsilon\,(x,z)\equiv\tilde{f}\left(x,\frac{z}{\varepsilon}\right)$$

where $z\in\Box_\varepsilon(x)$, $x\in\overline{\Omega}$. Moreover, for an arbitrary function $g\in C(\overline{\Omega})$ we have

$$(\varepsilon\to 0)\Rightarrow\left(\underset{x\in\overline{\Omega}}{\forall}g\big|_{\Box_\varepsilon(x)}\to g(x)\right)$$

We start with lagrangian $\mathcal{L}=\mathcal{L}(\cdot,\nabla w,w)\in HO_\delta^0(\Omega,\Box)$, where for the time being it is assumed that $w\in C^1\left(\overline{\Omega\times\Xi}\right)$. Due to the fact that lagrangian $\mathcal{L}$ is highly oscillating in Ω there exists for every $x\in\overline{\Omega}$ lagrangian $\tilde{\mathcal{L}}(x,\cdot,\nabla w,w)$ which is $\Box(x)$ periodic in R^m and constitutes a periodic approximation of lagrangian $\mathcal{L}$ in $\Box(x)$.

Let $h^A(\cdot)$, $A=1,...,N$ be a system of linear independent highly oscillating functions, $h^A(\cdot)\in HO_\delta^1(\Omega,\Box)$. It follows that there exist functions $\tilde{h}^A(x,\cdot)$, $A=1,...,N$

for every $x \in \overline{\Omega}$. Moreover, let $\tilde{h}_\varepsilon^A(x,\cdot)$, $A = 1,...,N$ be a family of functions given by

$$\tilde{h}_\varepsilon^A(x,z) \equiv \tilde{h}^A\left(x,\frac{z}{\varepsilon}\right), \quad z \in \Box_\varepsilon(x) \tag{1}$$

The fundamental assumption of the proposed procedure is that we introduce family of fields

$$w_\varepsilon(x,z,\xi) = u(z,\xi) + \varepsilon\tilde{h}_\varepsilon^A(x,z)v_A(z,\xi), \quad z \in \Box_\varepsilon(x), \ \xi \in \Xi \tag{2}$$

where summation over $A = 1,...,N$ holds. It is assumed that functions u, v_A are continuously bounded in $\overline{\Omega}$ together with their first derivatives. Formula (2) will be referred to as the consistent asymptotic decomposition of field $w(x,z,\xi)$, $z \in \Box(x)$, $\xi \in \Xi$.

Setting

$$\overline{\partial}\tilde{h}_\varepsilon^A(x,z) \equiv \frac{1}{\varepsilon}\partial\tilde{h}^A(x,\overline{z})\Big|_{\overline{z}=\frac{z}{\varepsilon}} \tag{3}$$

from formula (1) and (2) we obtain

$$\nabla w_\varepsilon(x,z,\xi) = \nabla u(z,\xi) + \overline{\partial}\tilde{h}_\varepsilon^A(x,z)v_A(z,\xi) + \varepsilon\tilde{h}_\varepsilon^A(x,z)\nabla v_A(z,\xi), \quad z \in \Box_\varepsilon(x) \tag{4}$$

for an arbitrary but fixed $x \in \overline{\Omega}$.

Here and subsequently in the proposed approach functions $u(\cdot)$, $v_A(\cdot)$, $A=1,\ldots,N$ are assumed to be independent of ε. This is the main difference between the asymptotic approach under consideration and approach which is used in the homogenisation theory [1, 4].

We have to keep in mind that by means of property of mean value, [1], functions $\tilde{h}_\varepsilon^A(x,z)$, $z \in \Box_\varepsilon(x)$, are weakly bounded and have under $\varepsilon \to 0$ weak limit. Taking into account $\varepsilon \to 0$, by virtue of $z \in \Box_\varepsilon(x)$, $x \in \overline{\Omega}$, we obtain

$$\begin{aligned}
&u(z,\xi) = u(x,\xi) + O(\varepsilon)\\
&\nabla u(z,\xi) = \nabla u(x,\xi) + O(\varepsilon)\\
&v_A(z,\xi) = v_A(x,\xi) + O(\varepsilon)\\
&\overline{\nabla} v_A(z,\xi) = \overline{\nabla} v_A(x,\xi) + O(\varepsilon)
\end{aligned} \tag{5}$$

By means of (5) we rewrite (2) and (4) in the form

$$
\begin{aligned}
w_\varepsilon(x,z,\xi) &= u(x,\xi) + O(\varepsilon) \\
\nabla w_\varepsilon(x,z,\xi) &= \nabla u(x,\xi) + \overline{\partial}\tilde{h}_\varepsilon^A(x,z)v_A(x,\xi) + O(\varepsilon)
\end{aligned}
\tag{6}
$$

Since subsequently we shall use a limit passage with ε to zero then terms $O(\varepsilon)$ in formula given above will be neglected. Let $\tilde{\mathcal{L}}_\varepsilon$ be a family of functions given by

$$
\tilde{\mathcal{L}}_\varepsilon = \tilde{\mathcal{L}}\left(x, \frac{z}{\varepsilon}, \nabla u(x,\xi), u(x,\xi), v_A(x,\xi)\right), \quad z \in \square_\varepsilon(x), \quad x \in \overline{\Omega}
$$

If $\varepsilon \to 0$ then $\tilde{\mathcal{L}}_\varepsilon$ as a function of z/ε, $z \in \square_\varepsilon(x)$, by means of the property of the mean value, [2], tends weakly in $L^\gamma_{loc}(R^m)$, $\gamma \geq 1$, to

$$
\begin{aligned}
&\mathcal{L}_0(x, \nabla u(x,\xi), u(x,\xi), v_A(x,\xi)) = \\
&= \frac{1}{|\square|}\int_{\square(x)} \tilde{\mathcal{L}}\left(x, z, \nabla u(x,\xi) + \partial\tilde{h}^A(x,z)v_A(x,\xi), u(x,\xi)\right)dz, \quad x \in \overline{\Omega}
\end{aligned}
\tag{7}
$$

Formula (7) will be referred to as the averaged form of lagrangian $\mathcal{L}$ under consistent asymptotic averaging.

3. Semiconsistent asymptotic averaging of integral functionals

In the previous section the asymptotic averaging of lagrangian $\mathcal{L} = \mathcal{L}(z, \nabla w, w)$, $z \in \overline{\Omega}$, was based on decompositions (2) of functions $w_\varepsilon(x,z,\xi)$, $z \in \square_\varepsilon(x)$, $\xi \in \Xi$ for every $x \in \overline{\Omega}$ and for every $\varepsilon = 1/n$, $n = 1,2,\ldots$. Now assume that $\mathcal{L}(\cdot)$ is independent on w, i.e. $\mathcal{L} = \mathcal{L}(z, \partial w, \overline{\nabla} w)$ and introduce two independent decompositions for $(\partial w)_\varepsilon(x,z,\xi)$ and $(\overline{\nabla} w)_\varepsilon(x,z,\xi)$, $z \in \square_\varepsilon(x)$, $\xi \in \Xi$, where $(\overline{\nabla} w)_\varepsilon \neq \overline{\nabla} w_\varepsilon$. The pertinent asymptotic modelling will be referred to as the semiconsistent asymptotic modelling while previously proposed procedure was called the consistent asymptotic modelling. We are to show that the semiconsistent modelling leads to more general equations than the consistent modelling, namely it makes it possible to describe phenomena depending on the size of cell $\square$.

For every $x \in \overline{\Omega}$, $\xi \in \Xi$ every $\varepsilon = 1/n$, $n = 1,2,\ldots$, and almost every $z \in \square_\varepsilon(x)$, assume

$$
(\partial w)_\varepsilon(x,z,\xi) = \partial u(z,\xi) + \varepsilon\partial\tilde{h}_\varepsilon^A(x,z)v_A(z,\xi)
\tag{8}
$$

and independently

$$(\overline{\nabla} w)_\varepsilon (x,z,\xi) = \overline{\nabla} u(z,\xi) + \tilde{h}_\varepsilon^A (x,z) \overline{\nabla} v_A (z,\xi) \tag{9}$$

Bearing in mind that functions $u(\cdot)$, $v_A(\cdot)$ are continuous and bounded together with the first derivatives and setting $\varepsilon \to 0$ for every $z \in \Box_\varepsilon(x)$, $x \in \overline{\Omega}$, we obtain

$$\begin{gathered} u(z,\xi) = u(x,\xi) + O(\varepsilon) \\ \nabla u(z,\xi) = \nabla u(x,\xi) + O(\varepsilon) \\ v_A(z,\xi) = v_A(x,\xi) + O(\varepsilon) \\ \overline{\nabla} v_A(z,\xi) = \overline{\nabla} v_A(x,\xi) + O(\varepsilon) \end{gathered} \tag{10}$$

By means of (3) and (10) we rewrite (8) and (9) in the form

$$\begin{gathered} (\partial w)_\varepsilon (x,z,\xi) = \partial u(x,\xi) + \overline{\partial} \tilde{h}_\varepsilon^A (x,z) v_A(x,\xi) + O(\varepsilon) \\ (\overline{\nabla} w)_\varepsilon (x,z,\xi) = \overline{\nabla} u(x,\xi) + \tilde{h}_\varepsilon^A (x,z) \overline{\nabla} v_A(x,\xi) + O(\varepsilon) \end{gathered} \tag{11}$$

After neglecting terms $O(\varepsilon)$ let us substitute the right-hand sides of (11) in the place of ∂w, $\overline{\nabla} w$, respectively, in lagrangian $\tilde{\mathcal{L}}_\varepsilon(x,z,\partial w, \overline{\nabla} w)$ where $z \in \Box_\varepsilon(x)$, $x \in \overline{\Omega}$. The obtained result has the form

$$\tilde{\mathcal{L}}_\varepsilon\left(x,z,\partial u(x,\xi) + \overline{\partial} \tilde{h}_\varepsilon^A (x,z) v_A(x,\xi), \overline{\nabla} u(x,\xi) + \tilde{h}_\varepsilon^A (x,z) \overline{\nabla} v_A(x,\xi)\right)$$

for a.e. $z \in \Box_\varepsilon(x)$ and every $x \in \overline{\Omega}$, $\xi \in \Xi$. By means of the property of the mean value, [2], there exists the weak limit passage for every $(z,\xi) \in \Omega \times \Xi$ such that if $\varepsilon \to 0$ then $\mathcal{L}_\varepsilon \rightharpoonup \mathcal{L}_0$ in $L^\gamma_{loc}(R^m)$ for $\gamma \geq 1$, where

$$\begin{gathered} \mathcal{L}_0\left(x, \nabla u(x,\xi), \overline{\nabla} v_A(x,\xi), v_A(x,\xi)\right) = \\ \frac{1}{|\Box|} \int_{\Box(x)} \tilde{\mathcal{L}}\left(x,z,\partial u(x,\xi) + \partial \tilde{h}^A(x,z) v_A(x,\xi), \overline{\nabla} u(x,\xi) + \tilde{h}^A(x,z) \overline{\nabla} v_A(x,\xi)\right) dz, x \in \overline{\Omega} \end{gathered} \tag{12}$$

Function $\mathcal{L}_0(x,\cdot)$, $x \in \overline{\Omega}$ represent the averaged form of lagrangian $\mathcal{L}(z,\partial w, \overline{\nabla} w)$, $z \in \Box(x)$, under semiconsistent asymptotic averaging.

4. Asymptotic modelling

In the framework of consistent asymptotic modelling we introduce the consistent asymptotic action functional defined by

$$\mathcal{A}_h^0(u,v_A)=\iint_{\Omega\,\Xi}\mathcal{L}_0(x,\nabla u,u,v_A)\,d\xi dx$$

where $\mathcal{L}_0$ is given by (7).
Under assumption that $\partial\mathcal{L}_0/\partial\nabla u$ is continuous, from the principle of stationary action for $\mathcal{A}_h^0(u,v_A)$, we obtain

$$\begin{aligned}&\nabla\cdot\frac{\partial\mathcal{L}_0}{\partial\nabla u}-\frac{\partial\mathcal{L}_0}{\partial u}=0\\&\frac{\partial\mathcal{L}_0}{\partial v_A}=0,\ \ A=1,\dots,N\end{aligned}\tag{13}$$

Let us recall that $\tilde{h}^A(x,\cdot)$, $A=1,\dots,N$, are periodic approximations of the postulated *a priori* fluctuation shape functions $h^A(\cdot)$ defined on $\overline{\Omega}$. Since function $w(\cdot)$ has to be uniquely defined in $\Omega\times\Xi$, we jump to the conclusion that $w(\cdot)$ has to take the form

$$w(z,\xi)=u(z,\xi)+h^A(z)v_A(z,\xi)\tag{14}$$

for every $(z,\xi)\in\Omega\times\Xi$. Equations (13) together with formula (14) represent the consistent asymptotic model of Euler-Lagrange equations derived from lagrangian $\mathcal{L}(z,\nabla w,w)$.

Now we pass to the semiconsistent asymptotic modelling. To this end let us introduce the semiconsistent asymptotic action functional defined by

$$\mathcal{A}_h^0(u,v_A)=\iint_{\Omega\,\Xi}\mathcal{L}_0(x,\nabla u,\overline{\nabla}v_A,v_A)\,d\xi dx$$

where $\mathcal{L}_0$ is given by (12).

Assuming that $\partial\mathcal{L}_0/\partial\nabla u$, $\partial\mathcal{L}_0/\partial\overline{\nabla}v_A$ are continuous, we obtain the model equations

$$\begin{aligned}&\nabla\cdot\frac{\partial\mathcal{L}_0}{\partial\nabla u}=0\\&\overline{\nabla}\cdot\frac{\partial\mathcal{L}_0}{\partial\overline{\nabla}v_A}-\frac{\partial\mathcal{L}_0}{\partial v_A}=0,\ \ A=1,\dots,N\end{aligned}\tag{15}$$

Let us observe that for $\varepsilon = 1$ formulae (8), (9) lead to formula

$$\nabla w(z,\xi) = \nabla\left(u(z,\xi) + \tilde{h}^A(x,z)v_A(z,\xi)\right)$$

which holds for every $z \in \square(x)$ and for an arbitrary but fixed $x \in \overline{\Omega}$. Subsequently we shall assume that

$$w(z,\xi) = u(z,\xi) + \tilde{h}^A(x,z)v_A(z,\xi) \tag{16}$$

for every $x \in \overline{\Omega}$ and a. e. $z \in \square(x)$. Equations (15) together with formula

$$w(z,\xi) = u(z,\xi) + h^A(z)v_A(z,\xi)$$

for every $(z,\xi) \in \Omega \times \Xi$ represents the semiconsistent asymptotic model of Euler--Lagrange equations derived from lagrangian $\mathcal{L}(z,\nabla w)$.

Conclusions

The proposed formal modelling can be applied to the formation of different mathematical models for the analysis of thermomechanical processes and phenomena in microheterogeneous solids and structures. The problems related to some applications of this approach will be studied in forthcoming papers.

References

[1] Jikov V., Kozlov S., Oleinik O., Homogenization of differential operators and integral functionals, Springer-Verlag, Berlin 1994.

[2] Woźniak C., Michalak B., Jędrysiak J. (eds), Thermomechanics of microheterogeneous solids and structures, Tolerance averaging approach, Wydawnictwo Politechniki Łódzkiej, Łódź 2008.

[3] Woźniak C. et al. (eds), Mathematical modelling and analysis in continuum mechanics of microstructured media, Wydawnictwo Politechniki Śląskiej, Gliwice (in the course of publication).

[4] Bensoussan A., Lions J.L., Papanicolau G., Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam 1978.