ON SOME EXTREMAL PROBLEM OF CHOSEN TWO-PARAMETRICAL FAMILIES OF REAL FUNCTIONS

Andrzej Wrzesień

Center of Mathematics and Physics, Technical University of Lodz, Poland
andrzej.wrzesień@p.lodz.pl

Abstract. In this paper we continue the investigation from [1]. There we have considered the case of the circle and the ellipse. Now we will take two-parametrical families of differentiable functions. For these families we determine *extremality coefficients*. In the last part of this paper we study the case of functions of two variables.

Introduction

In the region, $x \geq 0$, $y \geq 0$, let differentiable function $y = f(x)$ satisfy the following conditions:

$$\int_0^\infty f(x)dx < \infty \tag{$*$}$$

product $x \cdot f(x)$ has only one local maximum $\quad (**)$

Now, for any positive parameters a and b we define the following family of functions (deformation of basic function $f(x)$)

$$F_{a,b} \equiv \{f(x,a,b);\ \ x \geq 0,\ a > 0,\ b > 0,\ f(x,1,1) = f(x)\} \tag{1}$$

in such a way that every function of this family satisfies conditions $(*)$ and $(**)$.

Let $S(a,b)$ denote an area of the figure bounded by x - axis, y - axis and curve $y = f(x,a,b)$.

Obviously

$$S(a,b) = \int_0^\infty f(x,a,b)dx \tag{2}$$

Let $P_{a,b}^{[x \cdot y]}(x)$ denote a product of coordinates of the point (x, y) lying on the curve $y = f(x, a, b)$.

$$P_{a,b}^{[x \cdot y]}(x) = x \cdot f(x, a, b) \tag{3}$$

Finally, by $P_{\max}(a, b)$ we denote the value of local maximum of product (3) (see Fig. 1).

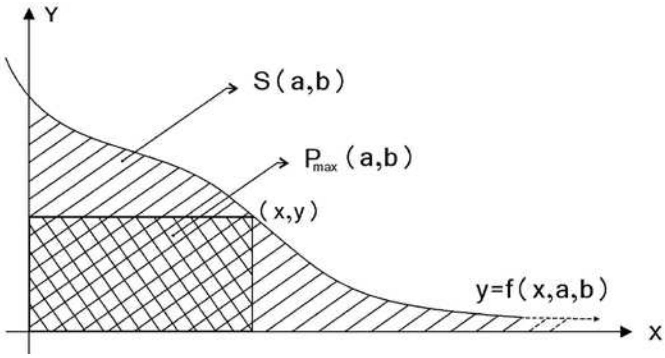

Fig. 1

According to the above notations we recall ([1]) primary definition, which will be used in our considerations.

Definition A quotient

$$K_e = \frac{S(a,b)}{P_{\max}(a,b)} \tag{4}$$

is said to be *extremality coefficient* of the family $F_{a,b}$.

In further considerations we will show that in many cases this coefficient is constant.

1. The case of a function of one variable

The first examples of our investigation (similarly like in [1]) are the families of functions considered in the finite interval.

Theorem 1.1 For the following families of polynomials:

$$F_{a,b}^{(1)} \equiv \left\{ y = b(1-\frac{x}{a}),\ \ x \in \langle 0,a \rangle,\ a>0,\ b>0 \right\}$$

$$F_{a,b}^{(2)} \equiv \left\{ y = b(1-\frac{x^2}{a^2}),\ \ x \in \langle 0,a \rangle,\ a>0,\ b>0 \right\}$$

$$F_{a,b}^{(3)} \equiv \left\{ y = b(1-\frac{x^3}{a^3}),\ \ x \in \langle 0,a \rangle,\ a>0,\ b>0 \right\}$$

extremality coefficients are respectively equal: $2, \sqrt{3}, \sqrt[3]{4}, \sqrt[4]{5}, ...$

Proof. (for the family $F_{a,b}^{(3)}$). Let us take any function of this family (see Fig. 2).

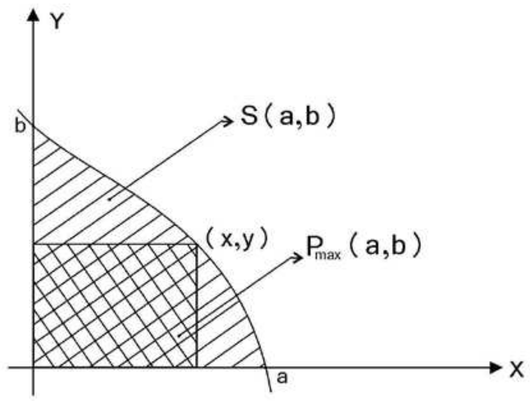

Fig. 2

According to notation (2) and (3)

$$S(a,b) = \int_0^a b\left(1-\frac{x^3}{a^3}\right)dx = \frac{3}{4}ab$$

$$P_{a,b}^{[x\cdot y]}(x) = x \cdot b\left(1-\frac{x^3}{a^3}\right)$$

The local maximum of the above function is attained at the point $x_0 = \frac{a}{\sqrt[3]{4}}$.
Hence

$$P_{\max}(a,b) = P_{a,b}^{[x \cdot y]}(x_0) = \frac{3}{4}\frac{ab}{\sqrt[3]{4}}$$

This gives $K_e = \sqrt[3]{4}$ □

This simple proof of other families is left to the reader. Now we will consider the behaviour of other families in interval $\langle 0,\infty \rangle$.

For the function $y = \frac{1}{x^2+1}$ we construct the following family

$$F_{a,b}^{(A)} \equiv \left\{ y = \frac{b}{x^2+a},\quad x \in \langle 0,\infty \rangle,\ a>0,\ b>0 \right\},$$ see Figure 3.

(generalization of Agnesi curve, $y = \frac{(2a)^3}{x^2+(2a)^2}$, $a>0$)

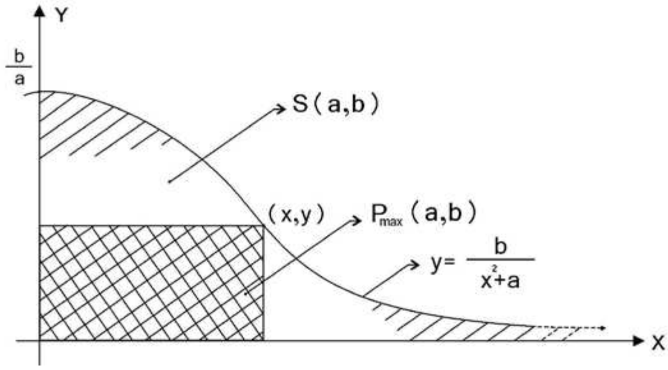

Fig. 3

Theorem 1.2 For the above family $K_e = \pi$.

Proof. Obviously, every function of this family satisfies conditions (∗) and (∗∗).
In this case we have

$$S(a,b) = \int_0^{\infty} \frac{b}{x^2+a}dx = \frac{\pi \cdot b}{2\sqrt{a}}$$

$$P_{a,b}^{[x,y]}(x) = \frac{xb}{x^2+a}$$

The derivative of the above function is equal zero at the point $x_0 = \sqrt{a}$. Therefore

$$P_{\max}(a,b)=P_{a,b}^{[x\cdot y]}(x_0)=\frac{b}{2\sqrt{a}}$$

Thus $K_e=\pi$ □.

We build the next family using the function $y=e^{-x^2}$.

$$F_{a,b}^{(G)}\equiv\left\{y=be^{-ax^2},\quad x\in\langle 0,\infty),\ a>0,\ b>0\right\}$$

(generalization of Gauss curve).

Theorem 1.3 For every function of family $F_{a,b}^{(G)}$

$$K_e=\sqrt{\frac{\pi e}{2}}$$

Proof. Since $\int_0^\infty e^{-\frac{1}{2}x^2}dx=\sqrt{\frac{\pi}{2}}$, we have $S(a,b)=\int_0^\infty be^{-ax^2}dx=\frac{b}{2}\sqrt{\pi a^{-1}}$. $P_{a,b}^{[x\cdot y]}(x)=bxe^{-ax^2}$, hence

$$P_{\max}(a,b)=P_{a,b}^{[x\cdot y]}\left(\frac{1}{\sqrt{2a}}\right)=\frac{b}{\sqrt{2ae}}$$

Therefore $K_e=\sqrt{\frac{\pi e}{2}}$ □.

Let us consider the exponential function $y=e^{-x}$, which also satisfies assumptions $(*)$ and $(**)$ and on the basis of this function we can construct the sequence of proper families.

Theorem 1.4 For the following families

$$F_{a,b}^{(0)}\equiv\left\{y=ae^{-bx},\quad x\in\langle 0,\infty),\quad a>0,\quad b>0\right\}$$
$$F_{a,b}^{(1)}\equiv\left\{y=axe^{-bx},\quad x\in\langle 0,\infty),\quad a>0,\quad b>0\right\}$$
$$\dots$$
$$F_{a,b}^{(n)}\equiv\left\{y=ax^ne^{-bx},\quad x\in\langle 0,\infty),\quad a>0,\quad b>0\right\}$$

extremality coefficients are respectively equal $\left(\frac{e}{1}\right)^1 0!$, $\left(\frac{e}{2}\right)^2 1!$, $\left(\frac{e}{3}\right)^3 2!$, . . . $\left(\frac{e}{n+1}\right)^{n+1} n!$

Not complicated proof of this theorem is left to the reader.

2. The case of a function of two variables

Similarly, like in Chapter 1 we consider differentiable function $z = f(x,y)$ in the region $x \geq 0,\ y \geq 0,\ z \geq 0.$

Our basic assumptions are the following:

$$\int_0^\infty \int_0^\infty f(x,y)dxdy < \infty \tag{$*$}$$

product $x \cdot y \cdot f(x,y)$ has only one local maximum. $(**)$

Now, using the mentioned function $f(x,y)$ and any positive parameters $a,b,c,$ we can construct the following family

$$F_{a,b,c} \equiv \{f(x,y,a,b,c);\ a>0, b>0, c>0,\ f(x,y,1,1,1) = f(x,y)\}$$

in such a way that every function of this family satisfies conditions $(*)$ and $(**)$.

In this case *extremality coefficient* is defined as follows

$$K_e = \frac{V(a,b,c)}{P_{\max}(a,b,c)}$$

where

$$\int_0^\infty \int_0^\infty f(x,y,a,b,c)dxdy = V(a,b,c)$$

$P_{\max}(a,b,c)$ denotes the maximum value of product
$x \cdot y \cdot f(x,y,a,b,c)$. (see Fig. 4)

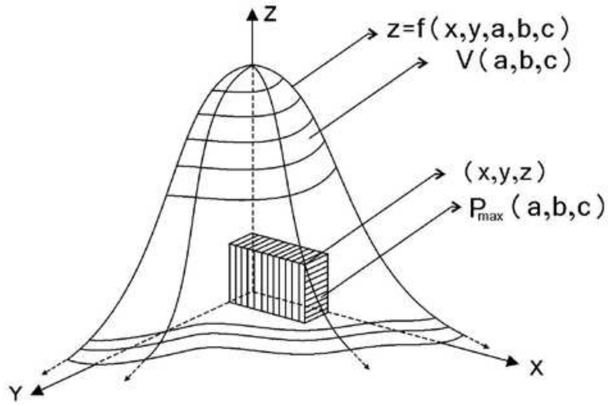

Fig. 4

As we know, many functions like plane and some typical surface of second order considered in the first octant satisfy conditions (∗) and (∗∗). Therefore, using three positive parameters a,b,c, we can construct the proper families of functions and investigate their *extremality coefficients*.

Theorem 2.1 For the following families:

$$F^{(1)}_{a,b,c} \equiv \left\{ z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right),\ \ x \in \langle 0,a\rangle,\ y \in \langle 0,b\rangle,\ z \in \langle 0,c\rangle \right\}$$

$$F^{(2)}_{a,b,c} \equiv \left\{ z = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),\ \ x \in \langle 0,a\rangle,\ y \in \langle 0,b\rangle,\ z \in \langle 0,c\rangle \right\}$$

$$F^{(3)}_{a,b,c} \equiv \left\{ z = c\left(1 - \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\right),\ \ x \in \langle 0,a\rangle,\ y \in \langle o,b\rangle,\ z \in \langle 0,c\rangle \right\}$$

$$F^{(4)}_{a,b,c} \equiv \left\{ z = c\left(\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\right),\ \ x \in \langle 0,a\rangle,\ y \in \langle 0,b\rangle,\ z \in \langle 0,c\rangle \right\}$$

extremality coefficients are equal, respectively:

$$K^{(1)}_e = \frac{9}{2},\ \ K^{(2)}_e = \pi,\ \ K^{(3)}_e = \frac{9}{16}\pi,\ \ K^{(4)}_e = \frac{\pi\sqrt{3}}{2}$$

Proof. (for the family $F^{(2)}_{a,b,c}$)

In this case

$$V(a,b,c) = \iint_D c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dxdy = \frac{\pi}{8}abc$$

where

$$D = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1,\quad x \in \langle 0, a\rangle,\; y \in \langle 0, b\rangle\right\}$$

Product $x \cdot y \cdot f(x,y,a,b,c) = c\left(xy - \frac{x^3 y}{a^2} - \frac{y^3 x}{b^2}\right)$ attains local maximum at the point $\left(\frac{a}{2}, \frac{b}{2}\right)$. Hence $P_{\max}(a,b,c) = \frac{1}{8}abc.$

This gives $K_e^{(1)} = \pi.$

Proofs of the remaining cases are similar.

From among many functions considered in the unlimited region, $x \ge 0,\; y \ge 0,$ which satisfy conditions $(*)$ and $(**)$, we take three further examples:

$$F_{a,b,c}^{(5)} \equiv \left\{z = \frac{a}{(x^2+b)(y^2+c)},\quad a>0,\; b>0,\; c>0\right\}$$

$$F_{a,b,c}^{(6)} \equiv \left\{z = axye^{-bx-cy},\quad a>0,\; b>0,\; c>0\right\}$$

$$F_{a,b,c}^{(7)} \equiv \left\{z = ae^{-bx^2-cy},\quad a>0,\; b>0,\; c>0\right\}$$

Using the well-known classical steps of differential and integral calculus we can proof the following theorem.

Theorem 2.2 For the above families, $F_{a,b,c}^{(5)}$, $F_{a,b,c}^{(6)}$ and $F_{a,b,c}^{(7)}$ *extremality coefficients* are equal, respectively:

$$K_e^{(5)} = \pi^2,\quad K_e^{(6)} = \frac{e^4}{16},\quad K_e^{(7)} = e\sqrt{\frac{\pi e}{2}}$$

Applications In many cases, specially when integration is not simple, we can use formula (4) and replace the value of integral by maximum value of proper area.

$$\int_0^\infty f(x,a,b)dx = K_e \cdot P_{\max}(a,b)$$

Example To evaluate integral $\int_0^\infty x^3 e^{-2x} dx$ we can write from the above:

$$\int_0^\infty x^3 e^{-2x} dx = K_e \cdot P_{\max}(1,2)$$

where $K_e = \frac{3! e^4}{4^4}$ (see Theorem 1.4),

(the whole family $\{ax^3 e^{-bx}\}$ has the same *extremality coefficient* equals $3!\left(\frac{e}{4}\right)^4$)

$P_{\max}(1,2)$ denotes maximum value of product $x \cdot x^3 e^{-2x}$. Since local maximum of this function attains at the point $x = 2$, then $P_{\max}(1,2) = 2^4 e^{-4}$. And consequently

$$\int_0^\infty x^3 e^{-2x} dx = \frac{3! e^4}{4^4} \cdot 2^4 \cdot e^{-4} = \frac{3!}{2^4}$$

Also for some double integrals we can use the same manner.

Remarks

At the end of this paper we would like to put some questions. The first one comes to mind at once.

1. How wide is the set of families possessing constant *extremality coefficient*?
2. The second question is whether for general case of two-parametrical function, there exist some criteria for possessing *extremality coefficient*.
3. For all investigated cases in the first Chapter (functions of one variable) *extremality coefficients* are not greater than π. In one case (Theorem 1.2) $K_e = \pi$. Therefore, the last question is whether $K_e \leq \pi$ for all two-parametrical functions[1].

References

[1] Wrzesień A., *In searching the number π*, Scientific Bulletin of Chełm 2007, 1, 237-239.

[1] The results of the paper were presented at the Second Forum of Polish Mathematicians, Częstochowa July, 1-4, 2008.