THE METHODOLOGY OF IMPROVEMENT OF CONSISTENT IN SAATY'S MATRIX JUDGMENTS

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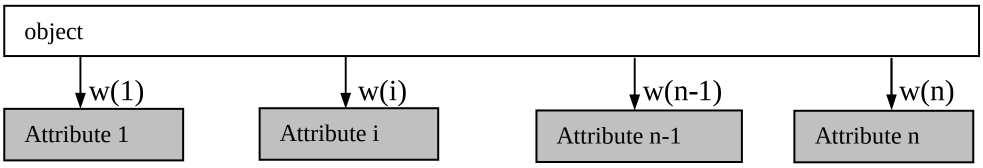
Abstract. Expert judgments presented as Saaty's matrix usually isn't consistent. In paper we propose simple iterative methods to improve level of consistent. This method based on feature of ideal consistent matrix. On the other hand, we understand that corrections (changes) of judgments shouldn't be carry out on a big scale (bigger then given determinant). Sometimes we try to limit corrective increments variance.

Introduction

Our aim is to present several variants of task as examples proving sensibility (based on next application profit [1-3]) of improve matrix of pairwise relative judgment (Saaty's matrix [4-7]). Consistent matrix of judgment give us at least credible structure of characteristic of attributes about analyzed object (firm, product, opinion, state of realization, condition of exploitation etc.) [8, 9]. We concentrate on two goals. The first is intuited "proving" that the transition of date (taking into account parameters of attributes) have hierarchically, iterative or recursive character. This entail a necessity of accumulation values of intermediate errors. Many level of nesting may cause finally aberration in a estimation or even decision. Traditionally presented variants of tasks regard constrains, changed variables and criteria [5, 10]. Additionally we want to close to real situation by consideration different form of uncertain date information, which can generate experts, standard assumptions, clients probes etc.

1. Classical situation relating to pairwise judgment of given object attributes

In real expert opinions (judgments) we use models basing on many attributes treating to one object (Fig. 1).



where w(i) - weight for i-th attribute

Fig. 1. Structure of estimated model with general weights

To get pairwise comparison we use classic table sizes m x m (where m - attribute number), which we create according to following rules (Table 1).

Table 1.

Pairwise comparison matrix

attribute	1	2	....	p	....	m-1	m
1	1	$\frac{w(1)}{w(2)}$	....	$\frac{w(1)}{w(p)}$	....	$\frac{w(1)}{w(m-1)}$	$\frac{w(1)}{w(m)}$
2	$\frac{w(2)}{w(1)}$	1	....	$\frac{w(2)}{w(p)}$	....	$\frac{w(2)}{w(m-1)}$	$\frac{w(2)}{w(m)}$
....	....	....	....	....	....	....	....
p	$\frac{w(p)}{w(1)}$	$\frac{w(p)}{w(2)}$	....	1	....	$\frac{w(p)}{w(m-1)}$	$\frac{w(p)}{w(m)}$
....	....	....	....	....	....	....	....
m-1	$\frac{w(m-1)}{w(1)}$	$\frac{w(m-1)}{w(2)}$	....	$\frac{w(m-1)}{w(p)}$	....	1	$\frac{w(m-1)}{w(m)}$
m	$\frac{w(m)}{w(1)}$	$\frac{w(m)}{w(2)}$	....	$\frac{w(m)}{w(p)}$	....	$\frac{w(m)}{w(m-1)}$	1

2. The process of evaluation level of consistent Saaty's model

Let's present functioning example - Saaty's model. The effect of arrangement of pairwise relative opinions was introduced as *example 1* in Table 2. Under diagonal judgment we have obviously inverted values a(i,j) = 1/a(j,i). On diagonal there are only ones: a(i,i)/a(i,i).

Generally we can use next formulas to estimate eigenvector and eigenvalue [1]:

$$w(p) = 1/m^* \sum_{j=1}^{m} \frac{a(p,j)}{\sum_{k=1}^{m} a(k,j)}$$

$$\lambda(i) = 1/w(i)\sum_{p=1}^{m} a(i,p) * w(p) = \frac{\sum_{p=1}^{m}(a(i,p)*(\sum_{j=1}^{m}(a(p,j)/\sum_{k=1}^{m}a(k,j)))}{\sum_{j=1}^{m}(a(i,j)/\sum_{k=1}^{m}a(k,j))}$$

where p - number of element in eigenvector.

The measurement of consistent (or inconsistent) of Saaty's matrix is represented by consistent and inconsistent coefficient (CI,CR) estimated on base λ_{max}:

$$CI = (\lambda_{max}-m)/(m-1)$$

$$\mathbf{CR = CI/R}$$

where R - random value from special tables [11].

Example 1

Table 2

Relative judgments are pairwise weights (pair: rows /column)

Date					
1,000	1,000	6,000	8,000	7,000	6,000
1,000	1,000	1,400	1,600	1,000	1,200
0,167	0,714	1,000	1,143	2,000	9,000
0,125	0,625	0,875	1,000	0,500	0,750
0,143	1,000	0,500	2,000	1,000	1,500
0,167	0,833	0,111	1,333	0,667	1,000
sums in columns					
2,601	5,173	9,886	15,076	12,167	19,450

We carry out the stage of preliminary standardization on base of sums in columns (Table 3).

Table 3

Standardized relative judgments from Table 2

0,384	0,193	0,607	0,531	0,575	0,308
0,384	0,193	0,142	0,106	0,082	0,062
0,064	0,138	0,101	0,076	0,164	0,463
0,048	0,121	0,089	0,066	0,041	0,039
0,055	0,193	0,051	0,133	0,082	0,077
0,064	0,161	0,011	0,088	0,055	0,051
sums in columns					
1,000	1,000	1,000	1,000	1,000	1,000

We sum up judgment in rows and again normalize (Tables 4, 5).

Tables 4, 5

The effect of summing up judgments in rows as well as the result of final standardization: eigenvector *w*

	sums in r		vector w
	2,599		0,433
	0,969		0,162
	1,006		0,168
	0,403		0,067
	0,591		0,098
	0,431		0,072
sum>	6,000	sum>	1

Using eigenvector we count the left hand side of equation $A * w = n * w$ (Table 6).

Table 6

Elements of vector $u = A*w$

	u=A*w
	3,259
	1,122
	1,276
	0,472
	0,648
	0,453
sum>	7,229

On basis $u = A*w$ we count elements of vector $\lambda(i)$ as well as its average value (eigenvalue) $\lambda_{max}.= \lambda_{aver}$ (Table 7).

Table 7

Elements of vector λ

	lamb=u/w
	7,523592
	6,943237
	7,606639
	7,02391
	6,580911
	6,299711
average>	6,996333

According to (1) and (2) we count the coefficients of consistent or inconsistent with help of equation $\lambda_{max}. = \lambda_{aver}$ (Table 8).

Table 8

Value of coefficients of consistent or inconsistent

consistent	inconsistent CI
0,160699	0,199266632

Basing on average value (eigenvalue) λ_{aver}, we can return to values of vector $u = \lambda_{aver} * w$ (Table 9).

Table 9

Effect of modification values of vector *u*

U=lmb_av*w
3,030743
1,130367
1,173319
0,470361
0,688898
0,502646

3. The idea of improvement level of consistent in Saaty's matrix judgments

The improvement level of consistent is connected with correction of judgments proposed by experts. In deterministic variant of date we may present our dealing with next scheme (Fig. 2):

M{a(1,1),...,a(i,j),...,a(m-1,m),...,a(m,m-1)},CI<>0 →

→ M{a(1,1),...,a(i,j)-d(i,j),...,a(m-1,m),...,a(m,m-1)},CI≈0

Fig. 2. The main idea of improvement matrix of judgment. Here M symbolizes Saaty's matrix of pairwise relative judgments, d(i,j) - correction increment

In above presented idea we realize correction on one or several judgment which should give us new matrix with better level of consistent in all structure of matrix. How showed experiences the best result give us sequential (iterative) correction on chosen judgment. The criteria of choosing judgment will be discussed in next part of our article. The main problem is how to evaluate values (increments) to realize procedure of correction(s). For it we can use only judgments from Saaty's matrix. We propose the method in which on the base of pair of given (or actual) judgment we can create another judgment.

The method of correction of relative deterministic judgments (assessments) for improvement of transitive profiles of Saaty's matrix ((det(A-Iλ)w)→0 lub λ→m)

Proposed method can be describe as follows:

1. Choice of row or column *k* with features of the largest credibility (the choice can have the subjective character).
2. Creation the matrix of corrective increments using next transformations:
   $\Delta(i,j) = a(i,j) - a(k,j)/a(k,i)$ (when the row was chosen $k \neq i$)
   and
   $\Delta(i,j) = a(i,j) - a(i,k)/a(j,k)$ (when the column was chosen $k \neq j$).
3. Choosing maximal (in absolute value sens) increments $\Delta_{max}(i_{max}, j_{max})$, where i_{max}, j_{max} - coordinates of location of the largest discrepancy.
4. Correction of relative weights:
   $aen(i_{max},j_{max}) = a(i_{max},j_{max}) - \Delta_{max}(i,j)$
5. Estimation of degree of consistent (CR) or inconsistent (CI) of the corrected matrix of relative weights:
   5.1. Normalizing value of estimation an(i,j) (eigenvector **w**)
   5.2. Calculation the value of elements of vector $\mathbf{u = A * w}$
   5.3. Estimation of the eigenvalue value $\boldsymbol{\lambda}$ and average of its elements λ_{aver}
   5.4. Calculation the coefficients of consistent CR or inconsistent CI
6. Checking requirements according with consistent or inconsistent (e.g. CI < 0,01)
7. Next iteration beginning from point 1.

Realization of point 1 in above described algorithm stays with problem to minimize its subjective character the task of selection of reference row or column vector: *k,* which thanks its structure will permit on minimizing the number of iteration.

There are graphic presentation of variants of method correction relative judgments. We want to extend illustration about possibilities of choosing correcting elements (rows and columns) (Tables 10).

Table 10

Rules of evaluating corrective increments

Variant A (Table 10a)
Zone over diagonal, correcting row k = 1, potentially corrected element a(i,j)

		a(k,i)		a(k,j)	
				a(i,j)	

- chosen zone
- correcting element
- corrected weight

aen(i,j) = a(k,i)/a(k,j), Δ(i,j) = a(i,j) - a(k,j)/a(k,i),

where *aen* - etalon level for comparison.

Variant B (Table 10b)
Zone over diagonal, correcting column k = m - 1, potentially corrected element a(i,j)

			a(i,j)	A(i,k)	
				A(j,k)	

aen(i,j) = a(j,k)/a(i,k), Δ(i,j) = a(i,j) - a(j,k)/a(i,k)

Variant C (Table 10c)
Zone over diagonal, correcting rows k = 2 and k = 3, potentially corrected elements $a(i_1,j_1)$ and $a(i_2,j_2)$

		$a(k,i_1)$	$a(k,j_1)$		
			$a(i_1,j_1)$	$a(k,i_2)$	$a(k,j_2)$
					$a(i_2,j_2)$

$aen(i_1,j_1) = a(k,i_1)/a(k,j_1),\ \Delta(i_1,j_1) = a(i_1,j_1) - a(k,j_1)/a(k,i_1)$
$aen(i_2,j_2) = a(k,i_2)/a(k,j_2),\ \Delta(i_2,j2) = a(i_2,j_2) - a(k,j_2)/a(k,i_2)$

Variant D (Table 10d)

Zone over diagonal, correcting row k = 1 and correcting column k = m = 6, potentially corrected elements $a(i_1,j_1)$ oraz $a(i_2,j_2)$

	$a(k,i_1)$			$a(k,j_1)$	
				$a(i_1,j_1)$	
				$a(i_2,j_2)$	$a(i_2,k)$
					$a(j_2,k)$

$aen(i_1,j_1) = a(k,i_1)/a(k,j_1),\ \Delta(i_1,j_1) = a(i_1,j_1) - a(k,j_1)/a(k,i_1)$
$aen(i_2,j_2) = a(j_2,k)/a(i_2,k),\ \Delta(i_2,j_2) = a(i_2,j_2) - a(j_2,k)/a(i_2,k)$

Variant E (Table 10e)

Zone under diagonal, correcting row k = 1, potentially corrected elements $a(i_1,j_1)$ and $a(i_2,j_2)$

$a(i_2,j_2)$					
		$a(i_1,j_1)$			
$a(k,j_2)$	$a(k,i_2)$	$a(k,j_1)$	$a(k,i_1)$		

$aen(i_1,j_1) = a(k,i_1)/a(k,j_1),\ \Delta(i_1,j_1) = a(i_1,j_1) - a(k,j_1)/a(k,i_1)$
$aen(i_2,j_2) = a(k,i_2)/a(k,j_2),\ \Delta(i_2,j_2) = a(i_2,j_2) - a(k,j_2)/a(k,i_2)$

Variant F (Table 10f)

Zone under diagonal, correcting row k = 1, potentially corrected elements $a(i_1,j_1)$ and $a(i_2,j_2)$

$a(j_2,k)$					
$a(i_2,k)$	$a(i_2,j_2)$				
	$a(i_1,j_1)$				
	$a(k,j_1)$		$a(k,i_1)$		

$aen(i_1,j_1) = a(k,i_1)/a(k,j_1),\ \Delta(i_1,j_1) = a(i_1,j_1) - a(k,j_1)/a(k,i_1)$
$aen(i_2,j_2) = a(j_2,k)/a(i_2,k),\ \Delta(i_2,j_2) = a(i_2,j_2) - a(j_2,k)/a(i_2,k)$

Method of averaged correction

From assumption of corrective method we inference that different rows and columns can be treated as corrective elements but only among these, which numbers are smaller (for rows) or larger (for columns) then the number of row (column) of corrected element (Tables 11a and 11b).

Table 11a

The example illustrating method of averaged correction based on corrective rows

x							
	X						
		x					
			x				
				X			
					x		
						x	
							x

$a^{(1)}(4,7) = a^{(0)}(1,7)/a^{(0)}(1,4)$
$a^{(2)}(4,7) = a^{(0)}(2,7)/a^{(0)}(2,4)$
$a^{(3)}(4,7) = a^{(0)}(3,7)/ a^{(0)}(3,4)$

where $a^{(r)}(i,j)$ - element corrected by r-th rows ($r < i$) placed in i-th row and j-th column

Table 11b

The example illustrating method of averaged correction based on corrective columns

X							
	x						
		x					
			x				
				X			
					x		
						x	
							x

$a^{(8)}(3,5) = a^{(0)}(3,8)/a^{(0)}(5,8)$
$a^{(7)}(3,5) = a^{(0)}(3,7)/a^{(0)}(5,7)$
$a^{(6)}(3,5) = a^{(0)}(3,6)/a^{(0)}(5,6)$

where
$a^{(c)}(i,j)$ - element corrected by c-th columns ($c > i$) placed in i-th row and j-th column

Generally

$a^{(1)}(i,j) = a^{(0)}(1,j)/\ a^{(0)}(1,i)$
$a^{(2)}(i,j) = a^{(0)}(2,j)/\ a^{(0)}(2,i)$
....
$a^{(i-1)}(4,7) = a^{(0)}(i-1,j)/a^{(0)}(i-1,i)$

$a^{(m)}(i,j) = a^{(0)}(i,m)/a^{(0)}(j,m)$
$a^{(m-1)}(i,j) = a^{(0)}(i,m-1)/a^{(0)}(j,m-1)$
....
$a^{(j+1)}(i,j) = a^{(0)}(i,j+1)/a^{(0)}(j,j+1)$

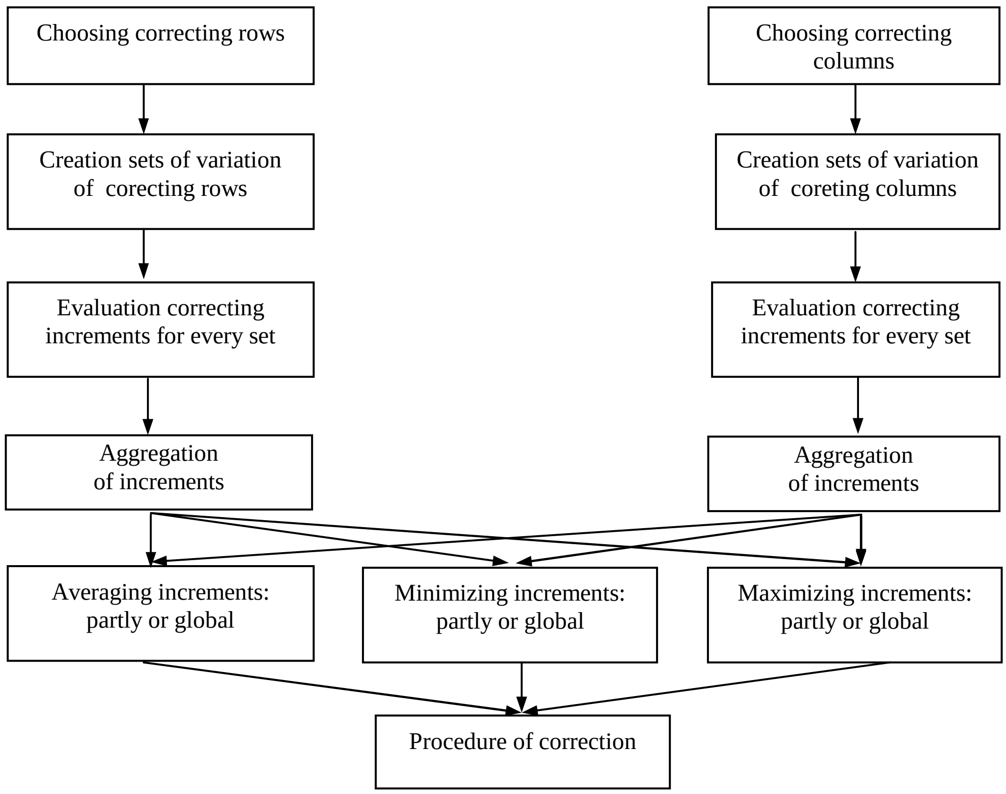


Fig. 3. Stages and variants of different form of choosing correcting increments

The conception of averaging relative pairwise judgments

$$a'(i,j) = 1/(i-1)\sum_{p=1}^{i-1} a^{(0)}(p,j)/a^{(0)}(p,i)$$

$$a''(i,j) = 1/(m\text{-}j) \sum_{p=j+1}^{m1} a^{(0)}(i,p)/a^{(0)}(j,p)$$

or

$$a''' = 1/(m\text{-}j\text{+}i\text{-}1)\{ \sum_{p=1}^{i-1} a^{(0)}(p,j)/a^{(0)}(p,i) + \sum_{p=j+1}^{m1} a^{(0)}(i,p)/a^{(0)}(j,p) \}$$

Estimation of correcting increments

$\Delta(i,j) = a^{(0)}(i,j) - a'(i,j)$

$\Delta(i,j) = a^{(0)}(i,j) - a''(i,j)$

or

$\Delta(i,j) = a^{(0)}(i,j) - a'''(i,j)$

This kind of graphical presentation can be useful to creation tasks chronology. Generally we may decide what combination of rows and columns we will use to correction process (to improving consistence of relative judgments. Set of possibilities is presented on diagram (Fig. 3).

Conclusions

Improving of relative pairwise judgment gives more consistent set of dependences referred to attribute validities. At the same time we destroy complex of experts opinions. Finally we obtain compromise which guaranties better level of credibility in reference to exploitation Saaty's matrix in practical applications. Important aim remains to work out modifications of iterative convention of improving method and invent some kind of heuristics recommended for typical situations. These situation treats to formulate global opinion about concrete objects, experts, firms, plans etc. Solution of this problem is on the way.

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