GREEN'S FUNCTIONS TO VIBRATION PROBLEMS OF BERNOULLI-EULER BEAMS WITH VARIABLE CROSS-SECTION BY A DIFFERENTIAL EQUATION SYSTEM

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Abstract. In this paper a method to determination of Green's functions of differential operators occurring in vibration problems of beams is presented. The fourth-order differential equation governing the vibration of a Bernoulli-Euler beam is as a first-order equation system rewritten. An approximate, analytic form of the solution to the boundary problem is obtained by using an Adomian decomposition method (ADM). The presented solution and numerical example deal the biharmonic equation with boundary conditions corresponding to the simply supported beam. Validation of the method has been proved to eigenvibration problem of a uniform beam.

Introduction

The Green's function method (GFM) has been applied successfully to many problems of mechanics and particularly to vibration problems of complex systems containing rods, beams and plates [1-3]. In any cases, use of the method leads to an exact solution of the considered problems. For instance, the Green's function method was used to solution of free vibration problem of stepped beams in reference [1] and cracked beams in [2].

However, some restriction in applications of the GFM are difficulties in determining of the Green's functions. The difficulties appear if the behavior of the considered system is governed by differential equations with variable coefficients. In these cases the solution often is available by using an approximate method only. Since the determination of the Green's function of differential operator consist in solution of a differential equation, in the situation of the equation with variable coefficients, the approximate method mostly must be applied.

In this paper an Adomian decomposition method is adopted to derive the Green's function corresponding to vibration problem of the simply supported beam. The approach concern the differential equation with variable coefficients which occur in Bernoulli-Euler theory of beam vibration. Validation of this analyt-ic-numerical method will be investigated in the case of an uniform beam when the exact form of the Green's function is known.

1. Formulation of the problem

The differential equation for small deflection of a beam is

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 w}{\partial t^2} = f(x,t) \tag{1}$$

where $w(x,t)$ denotes the beam deflection at cross-section x at time t, $f(x,t)$ is the transverse force acting on the beam, $EI(x)$ and $\rho A(x)$ are continuous functions which characterize the flexural stiffness and mass per unit length of the beam, respectively. In order to investigate the free vibration of the beam, one assumes that: $w(x,y) = W(x)e^{i\omega t}$, $f(x,y) = \tilde{f}(x)e^{i\omega t}$. The equation (1) can rewrite in the form:

$$\left(s(x)W''\right)'' - \omega^2 k(x)W = F(x) \tag{2}$$

where $F(x) = \tilde{f}(x)/EI(l/2)$, $k(x) = \rho A(x)/EI(l/2)$, $s(x) = EI(x)/EI(l/2)$, $0 \le x \le l$ and l denotes the length of the beam. The differential equation (2) is completed by two-point boundary conditions. In this study the boundary conditions corresponding to a simply supported beam are assumed: $W(0) = W''(0) = 0$, $W(l) = W''(l) = 0$. Searching a Green's function of the problem we assume that $F(x) = \delta(x-\xi)$, where $\delta(\cdot)$ is the Dirac delta function.

Introducing the functions: $y_1 = W$, $y_2 = W'$, $y_3 = s(x)W''$, $y_4 = W'''$, on the basis of equation (2), a system of differential equations of the first order is obtained:

$$\begin{cases} y_1' = & y_2 \\ y_2' = & \dfrac{1}{s(x)}y_3 \\ y_3' = & y_4 \\ y_4' = & \omega^2 k(x)y_1 + \delta(x-\xi) \end{cases} \tag{3}$$

In the considered case, the boundary conditions are:

$$y_1(0) = y_3(0) = 0, \quad y_1(l) = y_3(l) = 0 \tag{4}$$

2. Solution to the problem

After integration respect to x of the equations (3) one obtains:

$$\begin{cases} y_1 = & \int_0^x y_2(u)du \\ y_2 = & y_2(0) + \int_0^x \frac{1}{s(u)} y_3(u)du \\ y_3 = & \int_0^x y_4(u)du \\ y_4 = & y_4(0) + H(x-\xi) + \omega^2 \int_0^x k(u) y_1(u)du \end{cases} \tag{5}$$

We find an approximate solution of the system (5) by using Adomian decomposition method [4]. At first the functions y_i are assumed in the form of sums:

$$y_i \cong \sum_{n=1}^{N} y_{i,n}, \quad i = 1,\dots,4 \tag{6}$$

Next on the basis of the system (5) the following sequence of the equation systems is created:

$$\begin{cases} y_{1,0} = & 0 \\ y_{2,0} = & a \\ y_{3,0} = & 0 \\ y_{4,0} = & b + H(x-\xi) \end{cases} \qquad \begin{cases} y_{1,n+1} = & \int_0^x y_{2,n}(u)du \\ y_{2,n+1} = & \int_0^x \frac{1}{s(u)} y_{3,n}(u)du \\ y_{3,n+1} = & \int_0^x y_{4,n}(u)du \\ y_{4,n+1} = & \omega^2 \int_0^x k(u) y_{1,n}(u)du \end{cases}, \; n = 0, 1, 2, \dots \tag{7}$$

where $a = y_2(0)$, $b = y_4(0)$. The constants a and b are determined by using the boundary conditions (4b).

The functions $y_{i,n}$ can be determined for a given function $k(x)$ which form depends on the physical and geometrical properties of the beam. For uniform beam we have $k(x) = k = const$, $s(x) = 1$ for $0 \le x \le l$ and the functions $y_{i,n}$ are:

$$y_{1,1} = ax, \quad y_{2,1} = 0, \quad y_{3,1} = bx + (x-\xi)H(x-\xi), \quad y_{4,1} = 0$$

$$y_{1,2} = 0, \quad y_{2,2} = \frac{1}{2}\left(bx^2 + (x-\xi)^2 H(x-\xi)\right), \quad y_{3,2} = 0, \quad y_{4,2} = \frac{1}{2}akx^2\omega^2$$

$$y_{1,3}=\frac{1}{6}\left(b\,x^3+(x-\xi)^3\,H(x-\xi)\right),\quad y_{2,3}=0\,,\quad y_{3,3}=\frac{1}{6}akx^3\omega^2\,,\quad y_{4,3}=0$$

$$y_{1,4}=0\,,\quad y_{2,4}=\frac{1}{24}akx^4\omega^2\,,\quad y_{3,4}=0\,,\quad y_{4,4}=\frac{k\omega^2}{24}\left(b\,x^4+(x-\xi)^4\,H(x-\xi)\right)$$

$$y_{1,5}=\frac{1}{120}akx^5\omega^2\,,\quad y_{2,5}=0\,,\quad y_{3,5}=\frac{k\omega^2}{120}\left(b\,x^5+(x-\xi)^5\,H(x-\xi)\right),\quad y_{4,5}=0$$

$$y_{1,6}=0\,,\quad y_{2,6}=\frac{k\omega^2}{720}\left(b\,x^6+(x-\xi)^6\,H(x-\xi)\right),\quad y_{3,6}=0\,,\quad y_{4,6}=\frac{1}{720}ak^2x^6\omega^4$$

$$y_{1,7}=\frac{k\omega^2}{5040}\left(b\,x^7+(x-\xi)^7\,H(x-\xi)\right),\quad y_{2,7}=0\,,\quad y_{3,7}=\frac{1}{5040}ak^2x^7\omega^4\,,\quad y_{4,7}=0$$

. . . .

Using equation (6) we find the functions y_i. For instance the functions y_1 and y_3 are:

$$y_1=a\,x\left(1+\frac{k\omega^2}{120}x^4\right)+\frac{1}{6}b\,x^3\left(1+\frac{k\omega^2}{840}x^4\right)+\frac{1}{6}(x-\xi)^3\left[1+\frac{k\omega^2}{840}(x-\xi)^4\right]H(x-\xi) \tag{8}$$

$$y_3=\frac{1}{6}a\,k\,x^3\omega^2\left(1+\frac{k\omega^2}{840}x^4\right)+b\,x\left(1+\frac{k\omega^2}{120}x^4\right)+(x-\xi)\left[1+\frac{k\omega^2}{120}(x-\xi)^4\right]H(x-\xi) \tag{9}$$

Taken into account the functions in the boundary conditions (4b) an equation system is obtained. Solving the equation system respect to a and b, one obtains:

$$a=\frac{42\xi\,l_\xi\left(l-l_\xi\right)\left(120\left(840+k\,l^2\omega^2\left(l^2-6l_\xi^2\right)\right)+k\omega^2 l_\xi^4\left(120+k\,l^4\omega^2\right)\right)}{l\left(25401600-k\,l^4\omega^2\left(282240-k\,l^4\omega^2\left(84-k\,l^4\omega^2\right)\right)\right)} \tag{10}$$

$$b=\frac{-1764\,l_\xi\left(120+k\,l^4\omega^2\right)\left(120+k\,\omega^2 l_\xi^4\right)+k\,l^2\omega^2 l_\xi^3\left(840+k\,l^4\omega^2\right)\left(840+k\,\omega^2 l_\xi^4\right)}{l\left(25401600-k\,l^4\omega^2\left(282240-k\,l^4\omega^2\left(84-k\,l^4\omega^2\right)\right)\right)} \tag{11}$$

where $l_\xi=l-\xi$.

An approximate formula for the Green's function of the considered problem as a sum of $y_{1,n}$ is obtained:

$$G(x,\xi) \cong \sum_{n=1}^{N} y_{1,n}(x,\xi) \tag{12}$$

The function for N = 7 is presented by equation (8) with the a and b given by equations (10-11). The Green's function corresponding to the simply supported uniform beam as an exact solution of a boundary problem in the paper [3] has been derived.

3. Numerical example

The numerical calculations have been performed for the Green's function corresponding to a simply supported beam with the following data: l =1.0 m, k = 0,00363 (EI = 1,3772 10^{-4} Nm^2, ρA = 50 Ns^2/m^2). The results obtained for various values of x, ξ and ω, in Table 1 are presented. The values determined by using of the approximate formula (12) are compared with values calculated by using the exact formula of the Green's function.

The approximate and exact values of the Green's function as a function of the parameter ω are shown in Figure 1. The curves are obtained for $x = \xi = 0.5l$ and for the same parameters characterizing the beam which are used in Table 1. The results presented in the Table 1 as well as presented in Figure 1 show the small differences of the values for $\omega < 100$. From this it follows that the approx-imate formula (12) for the Green's function can be used in the range of ω, which includes the first eigenfrequency of the beam.

Table 1.

Values of the Green's function $G(x,\xi)$ for $\omega = 10$, $\omega = 50$, $\omega = 100$ and $\xi = 0.5$, $\xi = 0.75$; (a) approximate, eq. (12), (b) error, $\left|G_{exact}(x,\xi) - G_{appr}(x,\xi)\right|$.

x	ω = 10		ω = 50		ω = 100	
	(a)	(b)	(a)	(b)	(a)	(b)
ξ = 0.5						
0.1	0.006190	5.14×10^{-9}	0.006818	4.08×10^{-6}	0.009935	1.61×10^{-4}
0.2	0.011878	1.01×10^{-8}	0.013073	7.99×10^{-6}	0.019003	3.11×10^{-4}
0.3	0.016562	1.47×10^{-8}	0.018207	1.15×10^{-5}	0.026369	4.42×10^{-4}
0.4	0.019740	1.87×10^{-8}	0.021673	1.46×10^{-5}	0.031270	5.45×10^{-4}
0.5	0.020910	2.20×10^{-8}	0.022943	1.69×10^{-5}	0.033034	6.13×10^{-4}
ξ = 0.75						
0.1	0.003881	2.86×10^{-9}	0.004321	2.34×10^{-6}	0.006513	9.91×10^{-5}
0.2	0.007511	5.64×10^{-9}	0.008349	4.58×10^{-6}	0.012521	1.92×10^{-4}

0.3	0.010637	8.22×10^{-9}	0.011793	6.63×10^{-6}	0.017544	2.73×10^{-4}
0.4	0.013010	1.05×10^{-8}	0.014373	8.41×10^{-6}	0.021144	3.37×10^{-4}
0.5	0.014377	1.24×10^{-8}	0.015814	9.82×10^{-6}	0.022948	3.80×10^{-4}

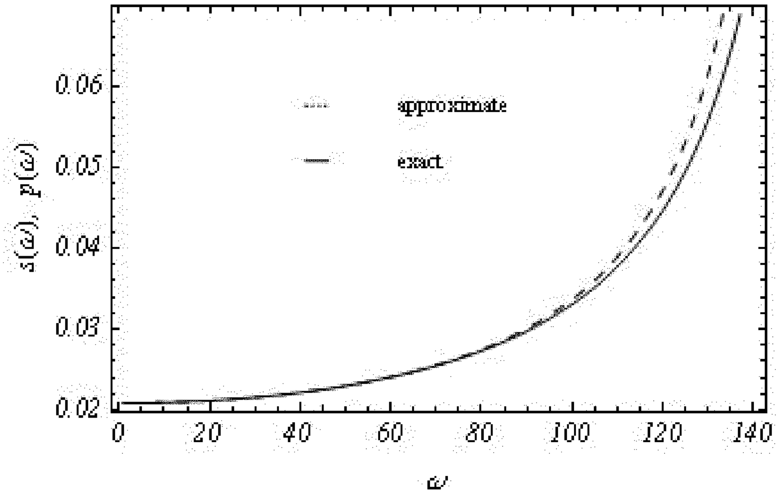

Fig. 1. Green's function for $x = \xi = 0.5l$ as a function of the parameter ω.

Conclusions

The Adomian decomposition method has been used to derive an approximate formula for the Green's function of a beam vibration problem. At the first step the fourth order differential equation as a system of equations is rewritten. Next a sequence of the differential equation systems is created. The solution of these sys-tems are members of sums which form the solution of the problem. The numerical investigations show that the obtained approximate formula for the Green's func-tion is valid in a range of the frequencies which includes the first eigenfrequency of the beam. Although the presented example deals the differential equation gov-erned the vibration of uniform beam, the approach can be used to derive the Green's functions which correspond to the vibration problem of the non-uniform beam.

References

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