ON SOME BLACKJACK TYPE OPTIMAL STOPPING PROBLEM

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Abstract. In the paper a class of optimal stopping problems which have some blackjack game features is considered. Both a value of the problem and an optimal stopping rule are found in some special case. Some examples and practical questions are considered as well.

Introduction

Blackjack (also known as “twenty-one") is the most popular casino table card game in the world. Blackjack is played on a points system that gives numeric values to every card in a single deck of playing cards. The cards are given to a player sequentially until he decides to stop (stand). His score is the sum of the values in hand. 21 is the best score one can achieve in the game, and players should be focusing *on getting as close to that number as possible without* **busting**. However, if a player’s cards exceed 21, then he has gone bust - the player *loses* and its bet is immediately taken by the dealer once this happens. The feature of the game we are interested in is the following*: the player with the highest total wins as long as it doesn't exceed a given limit number.*

In the paper we consider similar problem. Let $X_1, X_2, \ldots, X_N$ be a finite sequence of independent nonnegative random variables. A player observes sequentially the values and decides whether to stop or to continue. If he decides to stop at the moment k he gains a value $W(\sum_{i=1}^{k} X_i)$, where $W: R_+ \to R_+$ is a given nonnegative function. We assume that the function W is positive and increasing on the interval $(0, T]$ and is equal to zero for arguments greater than T. It means that the player obtains positive payoff which is the greater, the greater the sum $\sum_{i=1}^{k} X_i$ is, unless the sum exceeds a positive number T - a limit given in the problem (in blackjack game $T = 21$). If so, then the player gains 0. Our aim is to find a stopping rule which maximizes the expected payoff for a player.

Such a problem can be a model for various real world situations which can be observed in economics, finance, politics and social life. One specific problem of the type will be considered in detail in the sequel.

The problem outlined above belongs to so called optimal stopping problems which, historically, arose in the sequential analysis of statistical observations in early forties, see Wald [1]. In the next decade such problems were generalized to problems of pure stopping. In the 1960' papers of Chow and Robbins, see [2, 3], gave impetus to a rapid growth of the subject. The development of the theory and its applications were then summarized in a book [4]. A huge stimulus to the development of optimal stopping theory was also provided by the option pricing theory, developed in the 1970' in the theory of economics and finance, see [5]. Now the theory of optimal stopping is a scientific area of still growing interest to many applied theories in economics, finance etc. Recent developments in the theory and its applications are presented in [5-7].

1. Preliminaries - some general definitions and results

Before we solve our blackjack type optimal stopping problem we need to present some necessary formal definitions and fundamental results from the theory of optimal stopping. They may be found e.g. in [3, 4, 8].

Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Let $\mathfrak{F}_n$ denote the σ- algebra generated by the random variables $X_1, X_2, \ldots, X_n$ in an underlying probability space $(\Omega, \mathfrak{F}, \mathrm{P})$ *A stopping rule* is a random variable τ with values in a set of natural numbers such that $\{\tau = n\} \in \mathfrak{F}_n$ for $n = 1,2,\ldots$ and $\mathrm{P}(\tau < \infty) = 1$. Let $\mathsf{M}(n,N)$ be a class of all stopping rules τ such that $\mathrm{P}(n \le \tau \le N) = 1$. The class $\mathsf{M}(1,N)$ will be denoted $\mathsf{M}(N)$.

Let $(\mathsf{Y}_n, \mathfrak{F}_n)$, $n=1,2,\ldots$, be a homogenous Markov chain with values in a state space $(\mathsf{Y}, \mathfrak{B})$. Let $W : R \to R$ be a Borel measurable function which values $W(y)$ will be interpreted as the gain for a player when he stops the chain $(Y_n, \mathfrak{F}_n)$ at the state y. Assume that for a given state y and for a given stopping rule τ the expectation $\mathrm{E}(W(Y_\tau)|Y_1 = y)$ exists. Then it is natural to interpret the value - denoted by $\mathrm{E}_y W(Y_\tau)$ - as the mean gain corresponding to a chosen stopping rule τ. Let

$$V_N(y) = \sup_{\tau \in \mathsf{M}_W(N)} \mathrm{E}_y W(Y_\tau)$$

where $\mathsf{M}_W(N)$ is a set of all stopping rules belonging to $\mathsf{M}(N)$ for which the expectations $\mathrm{E}_y W(Y_\tau)$ exist for all $y \in \mathsf{Y}$ and are larger than $-\infty$. The function V_N is called a *value* of the problem of optimal stopping.

A stopping rule $\tau^* \in \mathsf{M}_W(N)$ satisfying the condition

$$\mathrm{E}_y W(Y_{\tau^*}) = V_N(y) \text{ for all } y \in \mathsf{Y}$$

is called an *optimal stopping rule*.

It is clear that the value $V_N(y)$ is the maximum possible mean gain to be obtained when the observation time is bounded by the number N. The following theorem, which can be found in [8], provides us with the solution of the optimal stopping problem in such a case. In order to state the theorem we need some additional definitions. Let $\mathfrak{B}$ denote a class of *all* Borel measurable functions W for which the expectations $E_yW(Y_1)$ exist for all $y \in \mathsf{Y}$. Let us define an operator Q operating on functions $W \in \mathfrak{B}$ by

$$QW(y) = \max\{W(y), E_yW(Y_1)\}$$

Theorem 1 (Shiryayev [8]). Assume that the gain function $W \in \mathfrak{B}$. Then:

i. $V_n(y) = Q^n(y)$, $n=1,2,\ldots$

ii. $V_n(y) = \max\{W(y), E_yV_{n-1}(Y_1)\}$, where $V_0(y) = W(y)$

iii. A stopping rule τ_n^* defined by

$$\tau_n^* = \min\{0 \le k \le n : V_{n-k}(y_m) = W(y_m)\}$$

is an optimal stopping rule in a class $\mathsf{M}_W(n)$

iv. If $E_y|W(Y_k)| < \infty$, for $k = 1,\ldots,n$, then the stopping rule τ_n^* is optimal in the class $\mathsf{M}(n)$

2. Problem statement and its solution

The problem we are to consider in detail can be formulated as follows. Let X_1, $X_2,\ldots, X_N$ be a sequence of independent random variables having the same exponential distribution with the density function

$$f(t;\lambda) = \lambda e^{-\lambda t}\mathbf{1}_{[0,\infty)}(t), \quad \lambda > 0 \tag{1}$$

Problem: The player observes the random sequence and decides whether to stop or to continue. If he decides to stop at the moment n he will gain $B\cdot(y+\sum_{i=1}^{n} X_i)$, $B > 0$, if the sum is not greater than T and will gain 0 otherwise. Find the optimal stopping rule and a value of the problem.

A given nonnegative real number y appearing in the above gain definition is an another characteristic of the problem and may be interpreted as an initial state of the process of observations.

It is easy to see that the problem is a special case of the general problem considered previously. Indeed, if we define a Markov chain $(Y_n, \mathfrak{F}_n)$ with

$$Y_n = y + \sum_{i=1}^{n} X_i, \; n = 1,\ldots, N \tag{2}$$

and $\mathfrak{F}_n$ being generated by the observations $X_1,\ldots, X_n$, and if the gain function W is given by the formula:

$$W(y;T) = B\, y\, \mathbf{1}_{(0,T]}(y) \tag{3}$$

with T and B being given real positive numbers, then we obtain the problem considered in the previous section. So, in order to solve the problem we apply Theorem 1. To do this first we need to find the form of $V_n(y) = Q^n(y)$, $n=1,2,\ldots,N$.
By definition of the operator Q we have for every $y \in (0,T]$:

$$QW(y) = \max\{W(y), E_y W(Y_1)\} = \max\{W(y), E_y W(y+X_1)\} =$$

$$\max\{W(y), \int_0^\infty W(y+x,T) f(x,\lambda)dx\} \overset{df}{=} \max\{W(y), I_1(y,T,\lambda)\}$$

For $y < T$ the function I_1 appearing above can be expressed as follows:

$$I_1(y,T,\lambda) = \int_0^\infty W(y+x,T) f(x,\lambda)dx = B\int_0^{T-y}(y+x)\lambda e^{-\lambda x}dx + \int_{T-y}^\infty 0\cdot\lambda e^{-\lambda x}dx =$$

$$\frac{B}{\lambda}(1-e^{(y-T)\lambda} - T\lambda e^{(y-T)\lambda} + y\lambda) \tag{4}$$

An exemplary typical graph of functions W and I_1 are presented on Figure 1.

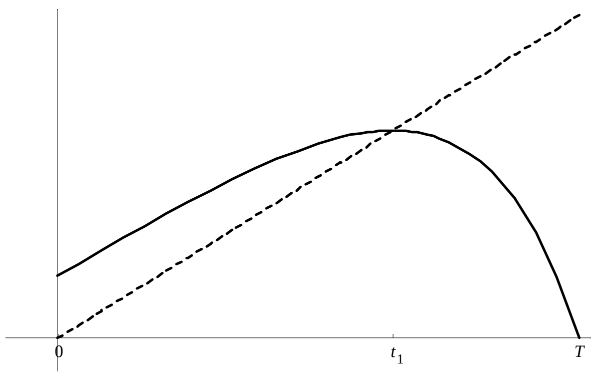

Fig. 1. Graphs of functions I_1 (continuous line) and W (dashed line)

It is easy to verify, that (for any given parameters B, T, λ characterizing our problem) a point t_1 for which the two functions have equal values is exactly the same as the point at which the function I_1 takes its only maximum on the interval $(0,T]$. Moreover the following conditions hold for every $T > 0$ and $\lambda > 0$:

$$I_1(y,T,\lambda) > W(y) \text{ for } y \in (0, t_1) \text{ and } I_1(y,T,\lambda) < W(y) \text{ for } y \in (t_1,\mathrm{T})$$

The value of t_1 depends only on parameters T, λ and is given by the formula:

$$t_1(T,\lambda) = T - \frac{1}{\lambda}\ln(1+T\lambda) \tag{5}$$

The function $V_1 = QW$ is the maximum of the two ones presented on Figure 1 and it follows from Theorem 1 that one step before the end of observations the player should continue the observations if he is at any state y which is less than t_1 and should stop otherwise. Obviously, the functions I_1, W and V_1 are equal to 0 for arguments greater than T.

Let $I_n(y,T,\lambda)$ denote the expectation $E_y V_{n-1}(y+X_1)$, $n = 1,\ldots,N$. Now, with the help of mathematical induction, we show that the following lemma is true.

Lemma. Let t_1 be given by the formula (5). Then for any natural number n and for every $T > 0$, $\lambda > 0$ the function I_n satisfies the following conditions:

i. $I_n(y,T,\lambda) > W(y)$ for $y \in (0, t_1)$

ii. $I_n(y,T,\lambda) < W(y)$ for $y \in (t_1,T]$

iii. $I_n(y,T,\lambda) = 0$ for $y > T$

Proof. It was already shown that conditions *i.*÷*iii*. hold for n=1. Now let us assume that conditions (*i*-*iii*) hold for $\mathbf{I}_{n-1}$. Then, by the definition of V_{n-1} and the induction assumption, we have for $y \in (0, t_1)$:

$$\begin{aligned} I_n(y,T,\lambda) &= \int_0^\infty V_{n-1}(y+x,T) f(x,\lambda)dx = \int_0^{t_1-y} I_{n-1}(y+x,T,\lambda) f(x,\lambda)dx + \\ &\int_{t_1-y}^{T-y} W(y+x) f(x,\lambda)dx + \int_{T-y}^{\infty} 0\cdot f(x,\lambda)dx \geq \\ &\int_0^{t_1-y} W(y+x) f(x,\lambda)dx + \int_{t_1-y}^{T-y} W(y+x) f(x,\lambda)dx = I_1(y,T,\lambda) > W(y) \end{aligned}$$

It implies that condition *i.* is satisfied.
On the other hand, when $y \in (t_1,T]$ we again obtain:

$$I_n(y,T,\lambda) = \int_0^\infty V_{n-1}(y+x,T) f(x,\lambda)dx = \int_0^{T-y} W(y+x) f(x,\lambda)dx = I_1(y,T,\lambda) < W(y)$$

and thus the condition *ii.* is satisfied. The condition *iii.* is obvious so the proof of the lemma is completed.
□

It follows from the lemma immediately that for n=1,…,N functions V_n have the form:

$$V_n(y) = I_n(y,T,\lambda)\cdot \mathbf{1}_{(0,t_1]}(y) + W(y)\cdot \mathbf{1}_{(t_1,T]}(y) \tag{6}$$

where t_1 is given by (5).

The following proposition provides us with the solution of our problem.

Proposition

Let us consider a sequence $X_1, X_2, \ldots, X_N$ of independent random variables with the same density functions given by (1). The optimal stopping rule for the problem of optimal stopping of the Markov chain (2) with the gain function (3) and initial state y is given by:

$$\tau_N^* = \min\{0 \le k \le N : Y_k = y + \sum_{i=1}^{k} X_k \ge t_1\}$$

with t_1 being given by the formula (5).

The value $V(y) = V_N(y)$ of the problem can be calculated for $y < t_1$ with the help of the following recursive equation

$$V_n(y, T, \lambda) = \int_0^{t_1 - y} V_{n-1}(y + x, T, \lambda)\lambda e^{-\lambda x} dx + \int_{t_1 - y}^{T - y} B \cdot (y + x)\lambda e^{-\lambda x} dx, \quad n=2,\ldots,N \qquad (7)$$

with the initial condition: $V_1(y, T, \lambda) = \dfrac{B}{\lambda}(1 - e^{(y-T)\lambda} - T\lambda e^{(y-T)\lambda} + y\lambda)$

We omit the proof of the Proposition because the results follow directly from Theorem 1, the Lemma and the formula (6).

The results stated in the Proposition imply that at any moment k the player should continue the observations if he is at any state y which is less than t_1 and should stop otherwise. Such a stopping rule maximizes his expected gain. The maximum what the player may expect to gain is $V(y)$.

3. Some examples and practical remarks

We can see that the recursive equation (7) involves integrating and one cannot be happy about it. We may notice however, that for any natural number n the functions V_n can be expressed in terms of elementary functions (involving e^x, $\ln x$, x^n) though the calculations are rather arduous, even for small numbers n. Fortunately, one may use computer with symbolic manipulation software such as *Mathematica*, *Maple*, *Maxima*, *Axiom*, etc., to obtain the form of the functions. For example we applied *Mathematica* 4.0 software to compute the form of the functions V_5, V_{10} and obtained the presented below *Mathematica* output (variables denoted y, T, λ, B have their previous meaning, symbol `Log` stands for natural logarithm denoted by ln in our paper).

The *Mathematica* output for V_5:

$$\begin{aligned}
&\frac{B}{24\lambda}\,(24\,e^{(-T+y)\lambda}\,(1+T\lambda)\,(T\lambda-\mathrm{Log}[1+T\lambda])+\\
&\quad e^{-T\lambda}\,(24\,e^{T\lambda}\,(5+y\lambda)-e^{y\lambda}\,(1+T\lambda)\,(120-96\,y\lambda+36\,y^2\lambda^2-8\,y^3\lambda^3+\\
&\qquad T^4\lambda^4+y^4\lambda^4-4\,T^3\lambda^3\,(-2+y\lambda)+6\,T^2\lambda^2\,(6-4\,y\lambda+y^2\lambda^2)-4\,T\lambda\,(-30+18\,y\lambda-6\,y^2\lambda^2+y^3\lambda^3))+\\
&\quad 4\,e^{y\lambda}\,(1+T\lambda)\,(30-18\,y\lambda+6\,y^2\lambda^2+T^3\lambda^3-y^3\lambda^3-3\,T^2\lambda^2\,(-2+y\lambda)+3\,T\lambda\,(6-4\,y\lambda+y^2\lambda^2))\,\mathrm{Log}[1+T\lambda]-\\
&\quad 6\,e^{y\lambda}\,(1+T\lambda)\,(6-4\,y\lambda+T^2\lambda^2+y^2\lambda^2-2\,T\lambda\,(-2+y\lambda))\,\mathrm{Log}[1+T\lambda]^2-\\
&\quad 4\,e^{y\lambda}\,(1+T\lambda)\,(-2-T\lambda+y\lambda)\,\mathrm{Log}[1+T\lambda]^3-e^{y\lambda}\,(1+T\lambda)\,\mathrm{Log}[1+T\lambda]^4))
\end{aligned}$$

The *Mathematica* output for V_{10}:

$$\begin{aligned}
&\frac{B}{362880\,\lambda}\,(e^{(-T+y)\lambda}\,(362880\,e^{(T-y)\lambda}\,(10+y\lambda)-\\
&\quad (1+T\lambda)\,(3628800-3265920\,y\lambda+1451520\,y^2\lambda^2-423360\,y^3\lambda^3+90720\,y^4\lambda^4-15120\,y^5\lambda^5+2016\,y^6\lambda^6-216\,y^7\lambda^7+\\
&\qquad 18\,y^8\lambda^8+T^9\lambda^9-y^9\lambda^9-9\,T^8\lambda^8\,(-2+y\lambda)+36\,T^7\lambda^7\,(6-4\,y\lambda+y^2\lambda^2)-84\,T^6\lambda^6\,(-24+18\,y\lambda-6\,y^2\lambda^2+y^3\lambda^3)+\\
&\qquad 126\,T^5\lambda^5\,(120-96\,y\lambda+36\,y^2\lambda^2-8\,y^3\lambda^3+y^4\lambda^4)-126\,T^4\lambda^4\,(-720+600\,y\lambda-240\,y^2\lambda^2+60\,y^3\lambda^3-10\,y^4\lambda^4+y^5\lambda^5)+\\
&\qquad 84\,T^3\lambda^3\,(5040-4320\,y\lambda+1800\,y^2\lambda^2-480\,y^3\lambda^3+90\,y^4\lambda^4-12\,y^5\lambda^5+y^6\lambda^6)-\\
&\qquad 36\,T^2\lambda^2\,(-40320+35280\,y\lambda-15120\,y^2\lambda^2+4200\,y^3\lambda^3-840\,y^4\lambda^4+126\,y^5\lambda^5-14\,y^6\lambda^6+y^7\lambda^7)+\\
&\qquad 9\,T\lambda\,(362880-322560\,y\lambda+141120\,y^2\lambda^2-40320\,y^3\lambda^3+8400\,y^4\lambda^4-1344\,y^5\lambda^5+168\,y^6\lambda^6-16\,y^7\lambda^7+y^8\lambda^8))+\\
&\quad 9\,(1+T\lambda)\,(362880-322560\,y\lambda+141120\,y^2\lambda^2-40320\,y^3\lambda^3+8400\,y^4\lambda^4-1344\,y^5\lambda^5+168\,y^6\lambda^6-16\,y^7\lambda^7+\\
&\qquad T^8\lambda^8+y^8\lambda^8-8\,T^7\lambda^7\,(-2+y\lambda)+28\,T^6\lambda^6\,(6-4\,y\lambda+y^2\lambda^2)-56\,T^5\lambda^5\,(-24+18\,y\lambda-6\,y^2\lambda^2+y^3\lambda^3)+\\
&\qquad 70\,T^4\lambda^4\,(120-96\,y\lambda+36\,y^2\lambda^2-8\,y^3\lambda^3+y^4\lambda^4)-56\,T^3\lambda^3\,(-720+600\,y\lambda-240\,y^2\lambda^2+60\,y^3\lambda^3-10\,y^4\lambda^4+y^5\lambda^5)+\\
&\qquad 28\,T^2\lambda^2\,(5040-4320\,y\lambda+1800\,y^2\lambda^2-480\,y^3\lambda^3+90\,y^4\lambda^4-12\,y^5\lambda^5+y^6\lambda^6)-\\
&\qquad 8\,T\lambda\,(-40320+35280\,y\lambda-15120\,y^2\lambda^2+4200\,y^3\lambda^3-840\,y^4\lambda^4+126\,y^5\lambda^5-14\,y^6\lambda^6+y^7\lambda^7))\,\mathrm{Log}[1+T\lambda]-\\
&\quad 36\,(1+T\lambda)\,(40320-35280\,y\lambda+15120\,y^2\lambda^2-4200\,y^3\lambda^3+840\,y^4\lambda^4-126\,y^5\lambda^5+14\,y^6\lambda^6+T^7\lambda^7-y^7\lambda^7-\\
&\qquad 7\,T^6\lambda^6\,(-2+y\lambda)+21\,T^5\lambda^5\,(6-4\,y\lambda+y^2\lambda^2)-35\,T^4\lambda^4\,(-24+18\,y\lambda-6\,y^2\lambda^2+y^3\lambda^3)+\\
&\qquad 35\,T^3\lambda^3\,(120-96\,y\lambda+36\,y^2\lambda^2-8\,y^3\lambda^3+y^4\lambda^4)-21\,T^2\lambda^2\,(-720+600\,y\lambda-240\,y^2\lambda^2+60\,y^3\lambda^3-10\,y^4\lambda^4+y^5\lambda^5)+\\
&\qquad 7\,T\lambda\,(5040-4320\,y\lambda+1800\,y^2\lambda^2-480\,y^3\lambda^3+90\,y^4\lambda^4-12\,y^5\lambda^5+y^6\lambda^6))\,\mathrm{Log}[1+T\lambda]^2+\\
&\quad 84\,(1+T\lambda)\,(5040-4320\,y\lambda+1800\,y^2\lambda^2-480\,y^3\lambda^3+90\,y^4\lambda^4-12\,y^5\lambda^5+T^6\lambda^6+y^6\lambda^6-6\,T^5\lambda^5\,(-2+y\lambda)+\\
&\qquad 15\,T^4\lambda^4\,(6-4\,y\lambda+y^2\lambda^2)-20\,T^3\lambda^3\,(-24+18\,y\lambda-6\,y^2\lambda^2+y^3\lambda^3)+15\,T^2\lambda^2\,(120-96\,y\lambda+36\,y^2\lambda^2-8\,y^3\lambda^3+y^4\lambda^4)-\\
&\qquad 6\,T\lambda\,(-720+600\,y\lambda-240\,y^2\lambda^2+60\,y^3\lambda^3-10\,y^4\lambda^4+y^5\lambda^5))\,\mathrm{Log}[1+T\lambda]^3-\\
&\quad 126\,(1+T\lambda)\,(720-600\,y\lambda+240\,y^2\lambda^2-60\,y^3\lambda^3+10\,y^4\lambda^4+T^5\lambda^5-y^5\lambda^5-5\,T^4\lambda^4\,(-2+y\lambda)+10\,T^3\lambda^3\,(6-4\,y\lambda+y^2\lambda^2)-\\
&\qquad 10\,T^2\lambda^2\,(-24+18\,y\lambda-6\,y^2\lambda^2+y^3\lambda^3)+5\,T\lambda\,(120-96\,y\lambda+36\,y^2\lambda^2-8\,y^3\lambda^3+y^4\lambda^4))\,\mathrm{Log}[1+T\lambda]^4+\\
&\quad 126\,(1+T\lambda)\,(120-96\,y\lambda+36\,y^2\lambda^2-8\,y^3\lambda^3+T^4\lambda^4+y^4\lambda^4-4\,T^3\lambda^3\,(-2+y\lambda)+6\,T^2\lambda^2\,(6-4\,y\lambda+y^2\lambda^2)-\\
&\qquad 4\,T\lambda\,(-24+18\,y\lambda-6\,y^2\lambda^2+y^3\lambda^3))\,\mathrm{Log}[1+T\lambda]^5-\\
&\quad 84\,(1+T\lambda)\,(24-18\,y\lambda+6\,y^2\lambda^2+T^3\lambda^3-y^3\lambda^3-3\,T^2\lambda^2\,(-2+y\lambda)+3\,T\lambda\,(6-4\,y\lambda+y^2\lambda^2))\,\mathrm{Log}[1+T\lambda]^6+\\
&\quad 36\,(1+T\lambda)\,(6-4\,y\lambda+T^2\lambda^2+y^2\lambda^2-2\,T\lambda\,(-2+y\lambda))\,\mathrm{Log}[1+T\lambda]^7-\\
&\quad 9\,(1+T\lambda)\,(2+T\lambda-y\lambda)\,\mathrm{Log}[1+T\lambda]^8+(1+T\lambda)\,\mathrm{Log}[1+T\lambda]^9))
\end{aligned}$$

Figure 2 shows the graphs of the functions V_1 – given by (4) –, V_5, V_{10} in a case where $T = 10$ and $B = \lambda = 1$.

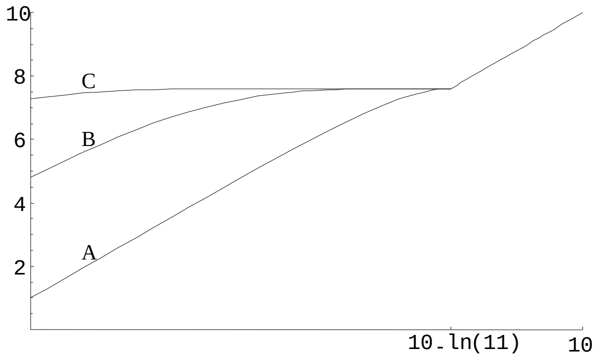

Fig. 2. Graphs of the values V_1, V_5, V_{10} as functions of an initial state of the process in the case where $T = 10$ and $B = \lambda = 1$ (plots A,B,C respectively)

We can easily compute the value $t_1(10,1)$ = 10 - ln(11) ≈7.602. One can also verify that in such a case

$$V_1(0,10,1) = 1-\frac{11}{e^{10}} \approx 0.9995,$$

$$V_5(0,10,1) = 5-\frac{11}{24e^{10}}(22680-7216\ln 11+876\ln^2 11-48\ln^3 11+\ln^4 11) \approx 4.796,$$

$$V_{10}(0,10,1) = 10+\frac{11}{362880e^{10}}(-10^{10}+6339784320\ln 11-1839784320\ln^2 11+ 319892160\ln^3 11-36630720\ln^4 11+2857680\ln^5 11-151536\ln^6 11+ 5256\ln^7 11-108\ln^8 11+\ln^9 11 \approx 7.279$$

Consequently, in the case where the limit value T equals 10 and the player have nothing starting the game - the initial state y equals 0 - he should continue his game until his total score (the sum of already observed values) exceeds 10-ln(11) ≈ ≈7.602. Applying this stopping rule he can expect to win (in average) about 0.9995 if he has got only one step to the end of observations, about 4.796 if he has 5 observations ahead, and about 7.279 if he has 10 observations before the end of the game. No other stopping rule can guarantee the player as much.

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