DETERMINATION OF TEMPERATURE FIELD IN DOMAIN OF COMPLEX SHAPE USING THE NURBS CURVES AND BEM

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Abstract. In the paper the temperature field determination in the domain of complex shape is presented. The boundary of the domain considered is described by the NURBS curves and the temperature field in this domain is calculated by means of the boundary element method. Such approach allows to determine the changes of temperature due to the local change of boundary configuration. In the final part of the paper the examples of computations are shown.

1. Boundary element method for Laplace equation

The steady state temperature field $T(x, y)$ in 2D domain is described by the Laplace equation

$$(x,y)\in\Omega \quad : \quad \nabla^2 T(x,y)=0 \tag{1}$$

supplemented by the boundary conditions

$$\begin{aligned}
&(x,y)\in \mathrm{C1}: \quad T(x,y)=T_b \\
&(x,y)\in \mathrm{C2}: \quad q(x,y)=-\lambda\,\mathbf{n}\cdot\nabla T(x,y)=q_b \\
&(x,y)\in \mathrm{C3}: \quad q(x,y)=-\lambda\,\mathbf{n}\cdot\nabla T(x,y)=\alpha\,[\,T(x,y)-T_a\,]
\end{aligned} \tag{2}$$

where λ [W/(mK)] is the thermal conductivity, T_b is known boundary temperature, $\mathbf{n}$ is the normal outward vector at the boundary point (x, y), q_b is given boundary heat flux, α is the heat transfer coefficient, T_a is the ambient temperature.

The integral equation for problem (1), (2) is following [1, 2]

$$\begin{aligned}
B(\xi,\eta)\,T(\xi,\eta)+\int_{\mathrm{C}} T^{*}(\xi,\eta,x,y)\,q(x,y)\,\mathrm{dC} = \\
\int_{\mathrm{C}} q^{*}(\xi,\eta,x,y)\,T(x,y)\,\mathrm{dC}
\end{aligned} \tag{3}$$

where (ξ, η) is the observation point (source point), if (ξ, η) ∈ C then $B(\xi, \eta)$ is the coefficient connected with the local shape of the boundary, if (ξ, η) ∈ Ω then $B(\xi, \eta) = 1$, $T^*(\xi, \eta, x, y)$ is the fundamental solution, while

$$q^*(\xi, \eta, x, y) = -\lambda\, \mathbf{n} \cdot \nabla T^*(\xi, \eta, x, y) \tag{4}$$

and

$$q(x, y) = -\lambda\, \mathbf{n} \cdot \nabla T(x, y) \tag{5}$$

Fundamental solution has the following form

$$T^*(\xi, \eta, x, y) = \frac{1}{2\pi\lambda} \ln \frac{1}{r} \tag{6}$$

where r is the distance between the points (ξ, η) and (x, y)

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \tag{7}$$

It should be pointed out that the function $T^*(\xi, \eta, x, y)$ fulfils the equation

$$\lambda \nabla^2 T^*(\xi, \eta, x, y) = -\delta(\xi, \eta, x, y) \tag{8}$$

where $\delta(\xi, \eta, x, y)$ is the Dirac function.

Heat flux resulting from the fundamental solution can be calculated analytically and then

$$q^*(\xi, \eta, x, y) = \frac{d}{2\pi r^2} \tag{9}$$

where

$$d = (x-\xi)\cos\alpha + (y-\eta)\cos\beta \tag{10}$$

while cosα, cosβ are the directional cosines of the boundary normal vector **n**.

To solve equation (3) the boundary C of the domain considered is divided into N boundary elements and then the approximation of equation (3) has the following form

$$\begin{gathered} B(\xi_i, \eta_i)\, T(\xi_i, \eta_i) + \sum_{j=1}^{N} \int_{C_j} T^*(\xi_i, \eta_i, x, y)\, q(x, y)\, dC_j = \\ \sum_{j=1}^{N} \int_{C_j} q^*(\xi_i, \eta_i, x, y)\, q(x, y)\, dC_j \end{gathered} \tag{11}$$

For constant boundary elements, namely

$$(x,y)\in \mathrm{C}_j: \quad \begin{cases} T(x,y)=T(x_j,y_j)=T_j \\ q(x,y)=q(x_j,y_j)=q_j \end{cases} \tag{12}$$

the equation (11) can be expressed as follows ($i = 1, 2, \ldots, N$)

$$\begin{gathered} \frac{1}{2}T_i+\sum_{j=1}^{N} q_j \int\limits_{\mathrm{C}_j} T^*(\xi_i,\eta_i,x,y)\ \mathrm{dC}_j = \\ \sum_{j=1}^{N} T_j \int\limits_{\mathrm{C}_j} q^*(\xi_i,\eta_i,x,y)\ \mathrm{dC}_j \end{gathered} \tag{13}$$

or

$$\sum_{j=1}^{N} G_{ij} q_j = \sum_{j=1}^{N} H_{ij} T_j \tag{14}$$

The solution of the system of equations (14), this means the values of temperatures or heat fluxes at the boundary nodes, allows to calculate the temperatures at internal nodes using the formula

$$T_i = \sum_{j=1}^{N} H_{ij} T_j - \sum_{j=1}^{N} G_{ij} q_j \tag{15}$$

2. Description of the boundary and its discretization

We consider the 2D domain Ω of complex shape. The segments of its boundary are described by the NURBS curves. A *n*-th degree NURBS curve is defined as [3]

$$C(t)=\frac{\sum_{j=0}^{r} N_{j,n}(t)\, w_j\, \mathbf{P}_j}{\sum_{k=0}^{r} N_{k,n}(t)\, w_k}, \qquad a \le t \le b \tag{16}$$

where $\mathbf{P}_j$ are the control points forming a control polygon, w_j are the weights and $N_{j,n}(t)$ are the B-spline basis functions

$$N_{j,0}(t)=\begin{cases} 1\ , & t_j \le t \le t_{j+1} \\ 0\ , & \text{otherwise} \end{cases}$$

$$N_{j,\,n}(t) = \frac{t - t_j}{t_{j+n} - t_j} N_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} N_{j+1,\,n-1}(t) \tag{17}$$

defined for the set of nodes

$$T = \left\{ a, ..., a, \; t_{n+1}, \; t_{n+2}, ..., \; t_{m-(n+1)}, b, .., b \right\} \tag{18}$$

at the same time the values *a* and *b* appear *n*+1 times. It should be pointed out that the number of control points equals *r*+1 and corresponds to the number of nonzero basis functions.

In Figure 1 the domain considered with marked control points $\mathbf{P}_0$ = (0, 0.04), $\mathbf{P}_1$ = (0.06, 0.04), $\mathbf{P}_2$ = (0.06, 0), $\mathbf{P}_3$ = (0.1, 0), $\mathbf{P}_4$ = (0.1, 0.1), $\mathbf{P}_5$ = (0, 0.1), $\mathbf{P}_6$ = (0.03, 0.08), $\mathbf{P}_7$ = (0.045, 0.08), $\mathbf{P}_8$ = (0.045, 0.07), $\mathbf{P}_9$ = (0.075, 0.05), $\mathbf{P}_{10}$ = (0.085, 0.05), $\mathbf{P}_{11}$ = (0.085, 0.035) is shown.

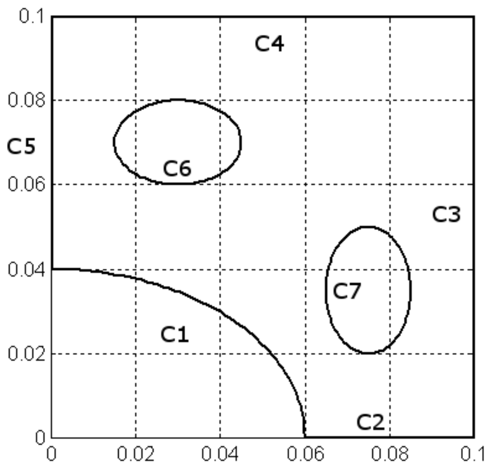

Fig. 1. Domain considered

The boundary is described by the following NURBS curves

C1: $C_1(t) = \dfrac{(1-t)^2 w_0 \mathrm{P}_0 + 2\,t(1-t) w_1 \mathrm{P}_1 + t^2 w_2 \mathrm{P}_2}{(1-t)^2 w_0 + 2\,t(1-t) w_1 + t^2 w_2}, \quad w_0 = w_1 = 1, \; w_2 = 2$

C2: $C_2(t) = (1-t)\mathrm{P}_2 + t\mathrm{P}_3,$

C3: $C_3(t) = (1-t)\mathrm{P}_3 + t\mathrm{P}_4,$

C4: $C_4(t) = (1-t)\mathrm{P}_4 + t\mathrm{P}_5,$

C5: $C_5(t) = (1-t)\mathrm{P}_5 + t\mathrm{P}_0,$

$$\text{C6}: \quad \begin{cases} C_6(t) = \dfrac{(1-t)^2 w_0 \text{P}_6 + 2t(1-t)w_1\text{P}_7 + t^2 w_2 \text{P}_8}{(1-t)^2 w_0 + 2t(1-t)w_1 + t^2 w_2}, \quad w_0 = w_1 = 1,\ w_2 = 2 \\ C_7(t) = C_6(t), \quad w_0 = 1, w_1 = -1,\ w_2 = 2 \end{cases}$$

$$\text{C7}: \quad \begin{cases} C_8(t) = \dfrac{(1-t)^2 w_0 \text{P}_9 + 2t(1-t)w_1\text{P}_{10} + t^2 w_2 \text{P}_{11}}{(1-t)^2 w_0 + 2t(1-t)w_1 + t^2 w_2}, \quad w_0 = w_1 = 1,\ w_2 = 2 \\ C_9(t) = C_8(t), \quad w_0 = 1, w_1 = -1,\ w_2 = 2 \end{cases}$$

The successive segments of the boundary are divided into N_1, N_2, N_3, N_4, N_5, N_6, N_7 boundary elements. In Figure 2 the boundary nodes under the assumption that $N_1 = 6$, $N_2 = 4$, $N_3 = 10$, $N_4 = 10$, $N_5 = 6$, $N_6 = 12$, $N_7 = 12$ are shown.

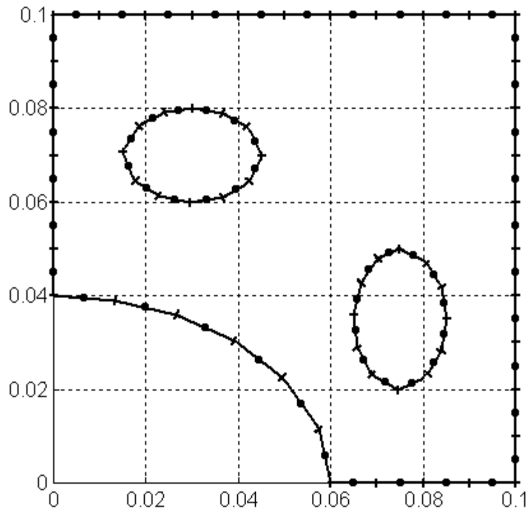

Fig. 2. Boundary nodes

3. Determination of temperature field

It is assumed that λ = 30 W/mK. The following boundary conditions on the successive segments of the boundary have been taken into account

$$\begin{array}{ll} (x,y) \in \text{C1}: & T(x,y) = 300 \\ (x,y) \in \text{C2} \cup \text{C3} \cup \text{C5} & q(x,y) = 0 \\ (x,y) \in \text{C4}: & T(x,y) = 20 \\ (x,y) \in \text{C6} \cup \text{C7}: & q(x,y) = 50\,(T(x,y) - 10) \end{array}$$

In Figure 3 the temperature distribution in the domain considered is shown.

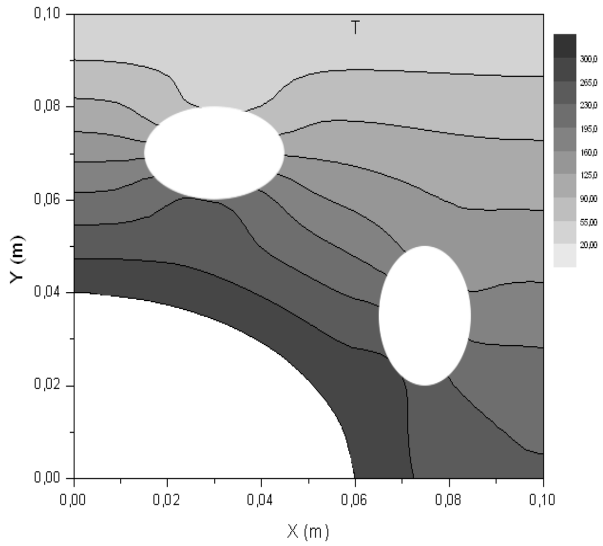

Fig. 3. Temperature distribution

It should be pointed out that the boundary element method coupled with the NURBS curves introduction is very useful, among others, in the case of shape sensitivity analysis applied in the heat transfer process modelling [4, 5]. These problems will be in future discussed.

References

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