ON THE TOLERANCE AVERAGING FOR DIFFERENTIAL OPERATORS WITH PERIODIC COEFFICIENTS

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Abstract. The aim of this contribution is to propose the tolerance averaging for differential operators with periodic coefficients. The averaging technique presented in this paper is based on proper limit passages with tolerance parameter to zero. This approach is a certain generalization of that presented in [1].

Introduction

The tolerance averaging of differential operators with periodic coefficients is based on the concept of slowly-varying and a special decomposition of unknown field. This concepts can be defined asymptotically by introducing to equation a small parameter λ. In the course of asymptotic homogenization parameter λ tends to zero but in every specific problem under consideration λ has to be treated as constant. In contrast to homogenization the tolerance averaging of differential operators is a non-asymptotic technique based on some physical hypotheses rather than on the formal analytical procedures. That is why we introduce physically reasonable non--asymptotic definitions of the slowly-varying functions. These definitions take into account the mathematical concept of tolerance and the physical idea of indiscernibility.

1. Basic notions and concepts

Let Ω be a regular region in R^n and m be a positive integer, $1 \le m \le n$. We define the basic cell $\square \equiv [-\lambda_1/2, \lambda_1/2] \times ... \times [-\lambda_n/2, \lambda_n/2]$ where $\lambda_i > 0$ for $i \le m$; $\lambda_i = 0$ for $i > m$. By λ we denote a diameter of $\square$. Moreover, we introduce denotation $\square(\mathbf{x}) \equiv \mathbf{x} + \square$, $\mathbf{x} \in R^n$ and assume that $\Omega_{\mathbf{x}} \equiv \square(\mathbf{x}) \cap \bar{\Omega}$ is the connected set for every $\mathbf{x} \in \bar{\Omega}$. For an arbitrary integer m, $1 \le m \le n$, we introduce gradient operators $\partial \equiv (\partial_1,...,\partial_m,0,...,0)$ and $\overline{\nabla} \equiv (0,...,0,\partial_{m+1},...,\partial_n)$ such that $\nabla = \partial + \overline{\nabla}$. Let $v \in C(\overline{\Omega})$, setting $v_{\mathbf{x}} = v|_{\Omega_{\mathbf{x}}}$, $\mathbf{x} \in \overline{\Omega}$, we introduce the following differences

$$\Delta v_{\mathbf{x}}(\mathbf{z}) \equiv v(\mathbf{z}) - v(\mathbf{x}) \quad \text{if} \quad \mathbf{z} \in \Omega_{\mathbf{x}}$$

$$\Delta(\nabla v_{\mathbf{x}}(\mathbf{z})) \equiv \nabla v(\mathbf{z}) - \nabla v(\mathbf{x}) \quad \text{if} \quad \mathbf{z} \in \Omega_{\mathbf{x}}$$

The tolerance averaging of differential operators with periodic coefficients is based on the concepts of slowly-varying and fluctuation shape functions. Now these concepts will be defined.

Definition 1

Function $v \in H^0(\Omega)$ will be called slowly varying function (with respect to the cell $\square$ and tolerance parameter δ) if for every $\mathbf{x} \in \Omega$

$$\|\Delta v_{\mathbf{x}}\|_{H^0(\Omega_{\mathbf{x}})} \leq \delta$$

The above condition will be written down in the form $v \in SV_\delta^0(\Omega;\square)$.

If for function $v \in H^1(\Omega)$ and for every $\mathbf{x} \in \Omega$ the following conditions hold

(i) $$\|\Delta v_{\mathbf{x}}\|_{H^0(\Omega_{\mathbf{x}})} \leq \delta$$

(ii) $$\|\Delta(\nabla v_{\mathbf{x}})\|_{(H^0(\Omega_{\mathbf{x}}))^n} \leq \delta$$

then we shall write $v \in SV_\delta^1(\Omega;\square)$, i.e. v is slowly varying together with its first gradient.

Definition 2

Periodic function $h \in H^1(\square)$ will be called fluctuation shape function, $h \in FS^1(\square)$, if

(i) $h(\mathbf{x}),\ \lambda\partial h(\mathbf{x}) \in O(\lambda)$ for a.e. $\mathbf{x} \in \Omega$

(ii) $\langle h \rangle = 0$, $\langle \nabla h \rangle = \mathbf{0}$

(iii) $\left(\forall v \in SV_\delta^1(\Omega;\square)\right)\left[\|h\partial v\|_{H^0(\Omega_{\mathbf{x}})} \leq \delta\right]$

Remark 1

If $v \in SV_\varepsilon^0(\Omega) \subset C^0(\Omega)$ then $(\forall \mathbf{x} \in \Omega)(\forall \mathbf{z} \in \Omega_{\mathbf{x}})[v(\mathbf{z}) \cong v(\mathbf{x})]$.

Remark 2

If $v \in SV_\varepsilon^1(\Omega) \subset C^1(\Omega)$ then

(i) $(\forall \mathbf{x} \in \Omega)(\forall \mathbf{z} \in \Omega_\mathbf{x})\left[v(\mathbf{z}) \cong v(\mathbf{x})\right]$

(ii) $(\forall \mathbf{x} \in \Omega)(\forall \mathbf{z} \in \Omega_\mathbf{x})\left[\nabla v(\mathbf{z}) \cong \nabla v(\mathbf{x})\right]$

(iii) $(\forall \mathbf{x} \in \Omega)(\forall \mathbf{z} \in \Omega_\mathbf{x})\left(\forall h \in FS^1(\square) \subset C^1(\Omega)\right)$

$$\left[\nabla\left(h(\mathbf{z})v(\mathbf{z})\right) \cong v(\mathbf{x})\partial h(\mathbf{z}) + h(\mathbf{z})\overline{\nabla} v(\mathbf{x})\right]$$

2. Fundamentals of averaging

In this Section we are to formulate averaging of a composite function by using limit passage with the tolerance parameter $\varepsilon \in (0, \delta]$ to zero. Let $v \in SV_\varepsilon^0(\Omega;\square)$. For every $\mathbf{x} \in \overline{\Omega}$ we define $v_\mathbf{x}^\varepsilon \in SV_\varepsilon^0(\Omega;\square)\Big|_{\Omega_\mathbf{x}}$ as a family of functions and $\mathbf{v}_\mathbf{x}^\varepsilon \in \left(SV_\varepsilon^0(\Omega;\square)\Big|_{\Omega_\mathbf{x}}\right)^n$ as a family of vector functions such that

(i) $v_\mathbf{x}^\varepsilon(\mathbf{z}) = v(\mathbf{x}) + O(\varepsilon), \quad \mathbf{z} \in \Omega_\mathbf{x}$

(ii) $\mathbf{v}_\mathbf{x}^\varepsilon(\mathbf{z}) = \nabla v(\mathbf{x}) + O(\varepsilon), \quad \mathbf{z} \in \Omega_\mathbf{x}$

(iii) $\nabla\left(h(\mathbf{z})v_\mathbf{x}^\varepsilon(\mathbf{z})\right) = \partial h(\mathbf{z})v(\mathbf{x}) + h(\mathbf{z})\overline{\nabla} v(\mathbf{x}) + O(\varepsilon), \quad \mathbf{z} \in \Omega_\mathbf{x}$

Hence if $\varepsilon \to 0$ then $\lim\limits_{\varepsilon \to 0} v_\mathbf{x}^\varepsilon(\mathbf{z}) = v(\mathbf{x})$ and $\lim\limits_{\varepsilon \to 0} \mathbf{v}_\mathbf{x}^\varepsilon(\mathbf{z}) = \nabla v(\mathbf{x})$.

Subsequently let $\varphi = \varphi\left(\mathbf{z}, w(\mathbf{z}), \nabla w(\mathbf{z})\right)$, $\mathbf{z} \in \Omega$, $w, \nabla w \in C(R)$ be a composite function such that $\varphi(\cdot, w, \nabla w) \in L^\alpha(\square)$ and $\varphi(\mathbf{z}, \cdot, \cdot) \in C\left(\overline{\Lambda}\right)$ for a.e. $\mathbf{z} \in \square$, $\Lambda \subset R^{n+1}$. The fundamental assumption imposed on field w in the framework of the tolerance averaging approach will be given in the form of the following *h*-decomposition

$$\begin{aligned} &w = w_h = v_0 + h^A v_A \\ &v_0, v_A \in SV_\delta^1(\Omega), \ h^A \in FS^1(\square), \ A = 1, \dots, N \end{aligned} \tag{1}$$

Under denotations $v \equiv (v_0, v_1, \dots, v_N)$, $h = \left(h^0, h^1, \dots, h^N\right)$, where $h^0 \equiv 1$ the aforementioned *h*-decomposition will be given in the form

$$w_h = h \cdot v \tag{2}$$

Definition 3

By the tolerance averaging of function φ under *h*-decomposition (2) we shall mean

$$\langle \varphi \rangle_h \left(v(\mathbf{x}), \tilde{\nabla} v(\mathbf{x}) \right) \equiv \frac{1}{|\square|} \int_{\square(\mathbf{x})} \lim_{\varepsilon \to 0} \varphi\left(\mathbf{z}, v_{\mathbf{x}}^{\varepsilon}(\mathbf{z}) \cdot h(\mathbf{z}), \nabla\left(v_{\mathbf{x}}^{\varepsilon}(\mathbf{z}) \cdot h(\mathbf{z}) \right) \right) d\mathbf{z}$$

where $\tilde{\nabla} v \equiv \left(\nabla v_0, \overline{\nabla} v_1, ..., \overline{\nabla} v_N \right)$.

Hence

$$\langle \varphi \rangle_h \left(v(\mathbf{x}), \tilde{\nabla} v(\mathbf{x}) \right) = \frac{1}{|\square|} \int_{\square} \varphi\left(\mathbf{y}, v(\mathbf{x}) \cdot h(\mathbf{y}), v(\mathbf{x}) \cdot \partial h(\mathbf{y}) + h(\mathbf{y}) \tilde{\nabla} v(\mathbf{x}) \right) d\mathbf{y} \tag{3}$$

It can be seen that function $\langle \varphi \rangle_h \in C\left(\overline{\Xi} \right)$ where Ξ is a bounded domain in $R^{N(1+n-m)+n+1}$.

3. Averaging of differential operators

The aim of this Section is to derive the tolerance averaging form of differential operator L and equation $Lw = f$ where $f = \nabla \cdot \mathbf{p}$, $\mathbf{p} = \mathbf{K} \nabla w$ and $w \in H^1(\Omega)$, $f \in L^2(\Omega)$, $K_{ij} \in L^2(\square)$, $i, j = 1, ..., n$. To this end we apply the *h*-decomposition setting

$$\begin{aligned} & w_h = v_K h^K, \quad K = 0,1,...,N \\ & h^0 \equiv 1, \quad h^A \in FS^1(\square), \quad A = 1,...,N \end{aligned} \tag{4}$$

Let $L_h^0 v = \langle f \rangle$, $L_h^K v = \langle h^K f \rangle$, $K = 1,...,N$ and $\tilde{\nabla} \mathbf{p}^K \equiv \nabla \mathbf{p}^0$ for $K = 0$, $\tilde{\nabla} \mathbf{p}^K \equiv \overline{\nabla} \mathbf{p}^K$ for $K = A = 1,...,N$. The vector operator $L_h = \left(L_h^0, L_h^1, ..., L_h^N \right)$ will be called the tolerance averaging of operator *L*.

Definition 4

Equation $L_h v = f_h$ defined by

$$
\begin{aligned}
& f^{K} \equiv \left\langle h^{K} f \right\rangle \\
& L_{h}^{K} v = f^{K}, \quad K = 0,1,..,N
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
& \tilde{\nabla} \cdot \mathbf{p}^{K} - r^{K} = f^{K} \\
& \mathbf{p}^{K} = \left\langle h^{K} \mathbf{K} \nabla \left(h^{L} v_{L} \right) \right\rangle_{h}
\end{aligned}
\tag{6}
$$

$$
r^{K} = \left\langle \partial h^{K} \mathbf{K} \nabla \left(h^{L} v_{L} \right) \right\rangle_{h}
\tag{7}
$$

is said to be the tolerance averaged equation for equation $Lw = f$ under decomposition (4).

Conclusions

The proposed formal modelling can be applied to the formation of different mathematical models for the analysis of thermomechanical processes and phenomena in microheterogeneous solids and structures. The problems related to some applications of this approach will be studied in forthcoming papers.

References

[1] Woźniak C., Wierzbicki E., Averaging Techniques in Thermomechanics of Composite Solids-Tolerance Averaging versus Homogenization, Częstochowa University Press, Częstochowa 2000.