THE POLYNOMIAL TENSOR INTERPOLATION

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Abstract. In this paper the tensor interpolation by polynomials of several variables is considered.

Introduction

The formulas of tensor interpolation by polynomials of several variables are the unknow in the interpolation methods ([1]). Using the Kronecker tensor product of matrices ([2, 3]) the polynomial tensor interpolation formula was given in this article.

1. The Kronecker product of matrices

The Kronecker product of two matrices \( A = \begin{bmatrix} a_{ij} \end{bmatrix} \) and \( B = \begin{bmatrix} b_{ij} \end{bmatrix} \) of degrees respectively \( m \) and \( n \) we define as a matrix given in block form as:

\[
A \otimes B = [a_{ij}B] = \begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1m}B \\
a_{21}B & a_{22}B & \cdots & a_{2m}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mm}B
\end{bmatrix}
\]

we denote the Kronecker product of \( A \) and \( B \) by \( A \otimes B \). The Kronecker product is also known as the direct product or the tensor product.

Some properties for Kronecker products of two matrices:

1. The matrices \( A \otimes B \) and \( B \otimes A \) are orthogonally similar, which means that square matrix \( U \) exists and \( B \otimes A = U'(A \otimes B)U \), \( U^TU = I \).

2. If \( A, B, C, D \) are square matrices such that the products \( AC \) and \( BD \) exist, then \((A \otimes B)(C \otimes D)\) exist and

\[
(A \otimes B)(C \otimes D) = AC \otimes BD
\]

(the “Mixed Product Rule”)
3. \((\alpha A) \otimes (\beta B) = \alpha \beta (A \otimes B)\)
4. \((A \otimes B)^t = A^t \otimes B^t\)
5. If \(A\) and \(B\) are invertible matrices, then \((A \otimes B)^{-1} = A^{-1} \otimes B^{-1}\)
6. \(\det(A \otimes B) = (\det A)^m (\det B)^n\)
7. The \((a \otimes b)_{rs}\) element of the matrix \(A \otimes B\) is given by the product

\[ (a \otimes b)_{rs} = a_{ij} b_{kl} \]

where \(r = (i-1)n + k\), \(s = (j-1)n + l\).

Next, we consider the quadratic matrices

\[
\begin{pmatrix}
A_1 & A_2 & \cdots & A_k
\end{pmatrix}
\]

and define the tensor product inductively:

\[
A_1 \otimes A_2 \otimes \cdots \otimes A_k = (A_1 \otimes A_2 \otimes \cdots \otimes A_{k-1}) \otimes A_k
\]

for \(k \geq 3\).

Some properties for Kronecker products:

1. The matrices \(A_{\sigma(1)} \otimes A_{\sigma(2)} \otimes \cdots \otimes A_{\sigma(k)}\) and \(A_1 \otimes A_2 \otimes \cdots \otimes A_k\), where \(\sigma\) is any determine permutation of numbers 1,...,\(k\), are orthogonaly similar. It means that square matrix \(U_\sigma\) exists and \(A_{\sigma(1)} \otimes A_{\sigma(2)} \otimes \cdots \otimes A_{\sigma(k)} = U_\sigma' (A_1 \otimes A_2 \otimes \cdots \otimes A_k) U_\sigma\) \(U_\sigma' U_\sigma = I\).

2. \((A_1 \otimes A_2 \otimes \cdots \otimes A_k) (B_1 \otimes B_2 \otimes \cdots \otimes B_k) = (A_1 B_1) \otimes (A_2 B_2) \otimes \cdots \otimes (A_k B_k)\) provided the dimensions of the matrices are such that the various expressions exist (the "Mixed Product Rule").

3. \((\alpha_1 A_1) \otimes (\alpha_2 A_2) \otimes \cdots \otimes (\alpha_k A_k) = \alpha_1 \alpha_2 \cdots \alpha_k (A_1 \otimes A_2 \otimes \cdots \otimes A_k)\)

4. \((A_1 \otimes A_2 \otimes \cdots \otimes A_k)^t = (A_1^t \otimes A_2^t \otimes \cdots \otimes A_k^t)\)

5. If \(A_1, \ldots, A_k\) are invertible matrices, then:

\[
(A_1 \otimes A_2 \otimes \cdots \otimes A_k)^{-1} = (A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_k^{-1})
\]

6. \(\det(A_1 \otimes A_2 \otimes \cdots \otimes A_k) = (\det A_1)^{n_1 \cdots \hat{n}_i \cdots n_k} (\det A_2)^{n_2 \cdots \hat{n}_2 \cdots n_k} \cdots (\det A_k)^{n_k \cdots \hat{n}_k \cdots n_k}\)

where \(\hat{n}_i\) is omission.

7. The \((a_1 \otimes a_2 \otimes \cdots \otimes a_k)_{ij}\) element of the matrix \(A_1 \otimes A_2 \otimes \cdots \otimes A_k\) is given by the product

\[
(a_1 \otimes a_2 \otimes \cdots \otimes a_k)_{ij} = (a_1)_{h_1} (a_2)_{h_2} \cdots (a_k)_{h_j}, \quad \text{where:}
\]

\[
i = (i_1 - 1) \hat{n}_1 n_2 \cdots n_k + (i_2 - 1) \hat{n}_2 n_3 \cdots n_k + \cdots + (i_k - 1) n_k + i_k
\]
2. One of the property for Vandermonde’s matrix

Consider the Vandermonde’s matrix:

\[ V_{p+1} = V_{p+1}(X_0, X_1, X_2, \ldots, X_p) = \begin{bmatrix}
1 & X_0 & \ldots & X_0^p \\
1 & X_1 & \ldots & X_1^p \\
\vdots & \vdots & \ddots & \vdots \\
1 & X_p & \ldots & X_p^p
\end{bmatrix} \]

of degree \( p + 1 \).

The algebraic complement of the matrix \( V_{p+1} \), has the form

\[ D_{ij} = (-1)^{i+j} \tau_{p-j}(X_0, \ldots, \hat{X}_i, \ldots, X_p) \det V_{p}(X_0, \ldots, \hat{X}_i, \ldots, X_p) \]

where \( \tau_{p-j}(X_0, \ldots, \hat{X}_i, \ldots, X_p) \) design the fundamental symmetric \( \tau_{p-j} \) polynomial of the rank \( p-j \) of variables \( X_0, \ldots, \hat{X}_i, \ldots, X_p \), and the symbol \( \hat{X}_i \) means omitting the variable \( X_i \) \( (\tau_0 = 0) \). Similarly for Vandermonde’s determinant \( \det V_{p} \).

This property we easily obtain on the one hand by evolving determinant

\[ \det V_{p+1} = \det \begin{bmatrix}
1 & X_0 & \ldots & X_0^p \\
1 & X_1 & \ldots & X_1^p \\
\vdots & \vdots & \ddots & \vdots \\
1 & X_p & \ldots & X_p^p
\end{bmatrix} = \prod_{0 \leq k \leq p} (X_i - X_k) \]

according to \( i \)-th row, and on the other hand by sorting its value according to \( X_i \) variable power.

In particular

\[ \frac{D_{ij}}{\det V_{p+1}} = (-1)^{i+j} \tau_{p-j}(X_0, \ldots, \hat{X}_i, \ldots, X_p) \]

\[ \Pi_i = \prod_{0 \leq k \leq p} \left( X_i - X_k \right) = (X_p - X_i) \ldots (X_{i+1} - X_i)(X_j - X_{i-1}) \ldots (X_j - X_0) \]

3. The polynomial tensor interpolation

The coefficients matrix \([a_{j_1 \ldots j_k}]\) of the polynomial interpolation

\[ W(X_1, \ldots, X_k) = \sum_{0 \leq j_1 \leq p_1, \ldots, 0 \leq j_k \leq p_k} a_{j_1 \ldots j_k} X_1^{j_1} \ldots X_k^{j_k} \]
are unknow. The results matrix $[(w_i)_j] = [w_{ij-1}]$ and the nodes matrix $[X_j] = [X_{i1} \ldots X_{ik}]$ are know. The coefficients of the polynomial are determined from the linear system

$$\left(\left[\begin{array}{c} X_{i1}^j \\
\vdots \\
X_{ikj}^j 
\end{array}\right] \otimes \ldots \otimes \left[\begin{array}{c} X_{i1}^j \\
\vdots \\
X_{ikj}^j 
\end{array}\right]\right)\mathbf{vec}[a_{j-1}] = \mathbf{vec}[w_{ij-1}]$$

where the nodes matrix is the Kronecker product of Vandermonde’s matrices

$$V_{p+1} = \left[\begin{array}{c} X_{i1}^j \\
\vdots \\
X_{ikj}^j 
\end{array}\right], \quad V_{p+1} = \left[\begin{array}{c} X_{i1}^j \\
\vdots \\
X_{ikj}^j 
\end{array}\right],$$

and the operator “vec” put coefficient matrix $[a_{j-1}]$ and results matrix $[w_{ij-1}]$ in columns, attributing multiindexes $j_1 \ldots j_k$ and $i_1 \ldots i_k$ properly positions

$$j = j_1(p_1 + 1)(p_2 + 1) \ldots (p_k + 1) + j_2(p_1 + 1)(p_2 + 1) \ldots (p_k + 1) + \ldots + j_{k-1}(p_k + 1) + j_k + 1$$

and

$$i = i_1(p_1 + 1)(p_2 + 1) \ldots (p_k + 1) + i_2(p_1 + 1)(p_2 + 1) \ldots (p_k + 1) + \ldots + i_{k-1}(p_k + 1) + i_k + 1$$

of the $a_{j-1}$ element in coefficients column and $w_{ij-1}$ element in results column. It means that we select ordering

$$00 \ldots 00, 00 \ldots 01, \ldots, 00 \ldots 0p_k, \ldots, 0 \ldots 1p_k, \ldots, 00 \ldots p_k, \ldots, 0p_1 \ldots p_k, \ldots, 0p_2 \ldots p_{k-1} \ldots p_k$$

with sequence shown above.

Then the searching coefficients column has a form

$$\mathbf{vec}[a_{j-1}] = \frac{1}{\det(X_{i1}^j)} \ldots \frac{1}{\det(X_{ikj}^j)} \mathbf{vec}[w_{ij-1}]$$

where: $V_i = V_{p+1} \left[\begin{array}{c} X_{i1}^j \\
\vdots \\
X_{ikj}^j 
\end{array}\right], V_j = V_{p+1} \left[\begin{array}{c} X_{i1}^j \\
\vdots \\
X_{ikj}^j 
\end{array}\right]$.

According to property 7’ we obtain the formula for coefficients

$$a_{j-1} = \sum_{\sigma(i) \leq p_1, \ldots, \sigma(i) \leq p_k} w_{ij-1} \frac{(D_{V_i} \cdot h_j)^{\sigma(i)}}{\det V_i} \cdot \frac{(D_{V_j} \cdot h_j)^{\sigma(i)}}{\det V_j}$$

and because fractions shown above has got known form then:
The polynomial tensor interpolation

\[ a_{-\ldots-\ldots} = \sum_{0 \leq q_{1} \ldots q_{k} \leq p_{1}} (-1)^{q_{1} + \ldots + q_{k}} \psi_{q_{1}+\ldots+q_{k}}^{(I_{1}+\ldots+I_{k})} X_{10} \ldots X_{1p_{1}} \]

\[ \tau_{\ldots-\ldots}^{q_{1}+\ldots+q_{k}}(X_{k0} \ldots X_{k_{p_{k}}}) \]

where: \( I^{+} = i_{1} + \ldots + i_{k} \), \( J^{+} = j_{1} + \ldots + j_{k} \) and

\[ \Pi_{i_{1}} = (X_{1p_{1}} - X_{1_{i_{1}}}) \ldots (X_{1_{i_{k-1}}} - X_{1_{i_{l}}}) (X_{1_{i_{l}}} - X_{1_{i_{l}}}) \ldots (X_{1_{i_{l}}} - X_{10}) \]

\[ \Pi_{k_{i_{l}}} = (X_{k_{p_{k}}} - X_{k_{i_{l}}}) \ldots (X_{k_{i_{k-1}}} - X_{k_{i_{l}}}) (X_{k_{i_{l}}} - X_{k_{i_{l}}}) \ldots (X_{k_{i_{l}}} - X_{k0}) \]

And now the polynomial coefficient we can obtain numerically.

References