THE POLYNOMIAL TENSOR INTERPOLATION

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Abstract. In this paper the tensor interpolation by polynomials of several variables is considered.

Introduction

The formulas of tensor interpolation by polynomials of several variables are the unknow in the interpolation methods ([1]). Using the Kronecker tensor product of matrices ([2, 3]) the polynomial tensor interpolation formula was given in tis article.

1. The Kronecker product of matrices

The Kronecker product of two matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ of degrees respectivly *m* and *n* we define as a matrix given in block form as:

$$A \otimes B = [a_{ij}B] = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mm}B \end{bmatrix}$$

we denote the Kronecker product of A and B by $A \otimes B$. The Kronecker product is also known as the *direct product* or the *tensor product*.

Some properties for Kronecker products of two matrices:

1. The matrices $A \otimes B$ and $B \otimes A$ are orthogonaly similar, which means that square matrix U exists and $B \otimes A = U^t (A \otimes B) U,\ \ U^t U = I$.
2. If *A*,*B*,*C*,*D* are square matrices such that the products AC and BD exist, then $(A \otimes B)(C \otimes D)$ exist and

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

(the "Mixed Product Rule")

3. $(\alpha A)\otimes(\beta B)=\alpha\beta(A\otimes B)$
4. $(A\otimes B)^t = A^t\otimes B^t$
5. If A and B are invertible matrices, then $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$
6. $\det(A\otimes B)=(\det A)^n(\det B)^m$
7. the $(a\otimes b)_{rs}$ element of the matrix $A\otimes B$ is given by the product

$$(a\otimes b)_{rs}=a_{ij}b_{kl}$$

where $r=(i-1)n+k$, $s=(j-1)n+l$.

Next, we consider the quadratic matrices $A_1=\left[(a_1)_{i_1 j_1}\right],\dots,A_k=\left[(a_k)_{i_k j_k}\right]$ of degrees respectivly $n_1,..,n_k$ and define the tensor product inductively:

$$A_1\otimes A_2\otimes\dots\otimes A_k=(A_1\otimes A_2\otimes\dots\otimes A_{k-1})\otimes A_k$$

for $k\geq 3$.

Some properties for Kronecker products:

1. The matrices $A_{\sigma(1)}\otimes A_{\sigma(2)}\otimes\dots\otimes A_{\sigma(k)}$ and $A_1\otimes A_2\otimes\dots\otimes A_k$, where σ is any determine permutation of numbers $1,\dots,k$, are orthogonaly similar. It means that square matrix U_σ exists and $A_{\sigma(1)}\otimes A_{\sigma(2)}\otimes\dots\otimes A_{\sigma(k)}=$ $=U_\sigma^t(A_1\otimes A_2\otimes\dots\otimes A_k)U_\sigma$, ${U_\sigma}^t U_\sigma=I$.
2. $(A_1\otimes A_2\otimes\dots\otimes A_k)(B_1\otimes B_2\otimes\dots\otimes B_k)=(A_1B_1)\otimes(A_2B_2)\otimes\dots\otimes(A_kB_k)$ provided the dimensions of the matrices are such that the various expressions exist (the "Mixed Product Rule").
3. $(\alpha_1A_1)\otimes(\alpha_2A_2)\otimes\dots\otimes(\alpha_kA_k)=\alpha_1\alpha_2\dots\alpha_k(A_1\otimes A_2\otimes\dots\otimes A_k)$
4. $(A_1\otimes A_2\otimes\dots\otimes A_k)^t=A_1^t\otimes A_2^t\otimes\dots\otimes A_k^t$
5. if $A_1,\dots,A_k$ are invertible matrices, then:
   $(A_1\otimes A_2\otimes\dots\otimes A_k)^{-1}=A_1^{-1}\otimes A_2^{-1}\otimes\dots\otimes A_k^{-1}$
6. $\det(A_1\otimes A_2\otimes\dots\otimes A_k)=(\det A_1)^{\hat{n}_1 n_2\dots n_k}(\det A_2)^{n_1\hat{n}_2\dots n_k}\dots(\det A_k)^{n_1 n_2\dots\hat{n}_k}$
   where $\hat{n}_j$ is omission.
7. the $(a_1\otimes a_2\otimes\dots\otimes a_k)_{ij}$ element of the matrix $A_1\otimes A_2\otimes\dots\otimes A_k$ is given by the product $(a_1\otimes a_2\otimes\dots\otimes a_k)_{ij}=(a_1)_{i_1j_1}(a_2)_{i_2j_2}\dots(a_k)_{i_kj_k}$, where:
   $i=(i_1-1)\hat{n}_1n_2\dots n_k+(i_2-1)\hat{n}_1\hat{n}_2\dots n_k+\dots+(i_{k-1}-1)n_k+i_k$

2. One of the property for Vandermonde's matrix

Consider the Vandermonde's matrix:

$$V_{p+1} = V_{p+1}\left(X_0, X_1, X_2, ..., X_p\right) = \begin{bmatrix} 1 & X_0 & \cdots & X_0^p \\ 1 & X_1 & \cdots & X_1^p \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_p & \cdots & X_p^p \end{bmatrix}$$

of degree $p+1$.

The algebraic complement of the matrix V_{p+1}, has the form

$$D_{ij} = (-1)^{i+j}\tau_{p-j}\left(X_0, ..., \hat{X}_i, ..., X_p\right)\det V_p\left(X_0, ..., \hat{X}_i, ..., X_p\right)$$

where $\tau_{p-j}\left(X_0, ..., \hat{X}_i, ..., X_p\right)$ design the fundamental symmetric τ_{p-j} polynomial of the rank $p-j$ of variables $X_0, ..., \hat{X}_i, ..., X_p$, and the symbol $\hat{X}_i$ means omitting the variable X_i ($\tau_0 = 0$). Similarly for Vandermonde's determinant $det V_p$.
This property we easly obtain on the one hand by evolving determinant

$$\det V_{p+1} = \det\begin{bmatrix} 1 & X_0 & \cdots & X_0^p \\ 1 & X_1 & \cdots & X_1^p \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_p & \cdots & X_p^p \end{bmatrix} = \prod_{0\le k<l\le p}\left(X_l - X_k\right)$$

according to ***i***-th row, and on the other hand by sorting its value according to X_i variable power.
In particular

$$\frac{D_{ij}}{\det V_{p+1}} = \frac{(-1)^{i+j}\tau_{p-j}\left(X_0, ..., \hat{X}_i, ..., X_p\right)}{\Pi_i}$$

where

$$\Pi_i = \prod_{\substack{0\le k<l\le p \\ k\ne i \text{ or } l\ne i}}\left(X_l - X_k\right) = \left(X_p - X_i\right)...\left(X_{i+1} - X_i\right)\left(X_i - X_{i-1}\right)...\left(X_i - X_0\right)$$

3. The polynomial tensor interpolation

The coefficients matrix $\left[a_{j_1 \cdots j_k}\right]$ of the polynomial interpolation

$$W\left(X_1, ..., X_k\right) = \sum_{0\le j_1\le p_1, ..., 0\le j_k\le p_k} a_{j_1\cdots j_k} X_1^{j_1} ... X_k^{j_k}$$

are unknow. The results matrix $\left[(w_1)_{i_1}...(w_k)_{i_k}\right]=\left[w_{i_1...i_k}\right]$ and the nodes matrix $\left[(X_1)_{i_1}\times...\times(X_k)_{i_k}\right]=\left[X_{1i_1}...X_{ki_k}\right]$ are know. The coefficients of the polynomial are determined from the linear system

$$\left(\left[X_{i_1}^{j_1}\right]\otimes...\otimes\left[X_{i_k}^{j_k}\right]\right)vec\left[a_{j_1...j_k}\right]=vec\left[w_{i_1...i_k}\right]$$

where the nodes matrix is the Kronecker product of Vandermonde's matrices $V_{p_1+1}=\left[X_{1i_1}^{j_1}\right],...,V_{p_k+1}=\left[X_{ki_k}^{j_k}\right]$, and the operator "vec" put coefficient matrix $\left[a_{j_1...j_k}\right]$ and results matrix $\left[w_{i_1...i_k}\right]$ in columns, attributing multiindexes $j_1...j_k$ and $i_1...i_k$ properly positions

$$j=j_1(\hat{p}_1+1)(p_2+1)...(p_k+1)+j_2(\hat{p}_1+1)(\hat{p}_2+1)...(p_k+1)+ \\ ...+j_{k-1}(p_k+1)+j_k+1$$

and

$$i=i_1(\hat{p}_1+1)(p_2+1)...(p_k+1)+i_2(\hat{p}_1+1)(\hat{p}_2+1)...(p_k+1)+ \\ ...+i_{k-1}(p_k+1)+i_k+1$$

of the $a_{j_1...j_k}$ element in coefficients column and $w_{i_1...i_k}$ element in results column. It means that we select ordering

$$00...00, 00...01, ..., 00...0p_k, 0...1p_k, ..., 00...p_{k-1}p_k, \\ ......, 0p_2...p_{k-1}p_k, ..., p_1...p_{k-1}p_k$$

with sequence shown above.
Then the searching coefficients column has a form

$$vec\left[a_{j_1...j_k}\right]=\frac{1}{\det\left[X_{1i_1}^{j_1}\right]}\cdots\frac{1}{\det\left[X_{ki_k}^{j_k}\right]}\left(D_{V_1}\otimes...\otimes D_{V_k}\right)^t vec\left[w_{i_1...i_k}\right]$$

where: $V_1=V_{p_1+1}=\left[X_{1i_1}^{j_1}\right],...,V_k=V_{p_k+1}=\left[X_{ki_k}^{j_k}\right]$.
According to property 7' we obtain the formula for coefficients

$$a_{j_1...j_k}=\sum\nolimits_{0\le i_1\le p_1,...,0\le i_k\le p_k} w_{i_1...i_k}\frac{\left(D_{V_1}\right)_{i_1j_1}}{\det V_1}\cdots\frac{\left(D_{V_k}\right)_{i_kj_k}}{\det V_k}$$

and because fractions shown above has got known form then:

$$a_{j_1 \dots j_k} = \sum_{0 \le i_1 \le p_1, \dots, 0 \le i_k \le p_k} (-1)^{I^+ + J^+} w_{i_1 \dots i_k} \frac{\tau_{p_1 - j_1}\left(X_{10}, \dots, \hat{X}_{1i_1}, \dots, X_{1p_1}\right)}{\Pi_{1i_1}}$$

$$\dots \frac{\tau_{p_k - j_k}\left(X_{k0}, \dots, \hat{X}_{ki_k}, \dots, X_{kp_k}\right)}{\Pi_{ki_k}}$$

where: $I^+ = i_1 + \dots + i_k$, $J^+ = j_1 + \dots + j_k$ and

$$\Pi_{1i_1} = \left(X_{1p_1} - X_{1i_1}\right) \dots \left(X_{1i_1+1} - X_{1i_1}\right)\left(X_{1i_1} - X_{1i_1-1}\right) \dots \left(X_{1i_1} - X_{10}\right)$$

$$\Pi_{ki_k} = \left(X_{kp_k} - X_{ki_k}\right) \dots \left(X_{ki_k+1} - X_{ki_k}\right)\left(X_{ki_k} - X_{ki_k-1}\right) \dots \left(X_{ki_k} - X_{k0}\right)$$

And now the polynomial coefficient we can obtain numerically.

References

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