# ON THE COMATIBILITY OF THE TANGENCY RELATIONS OF RECTIFIABLE ARCS

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**Abstract.** In this paper the problem of the compatibility of the tangency relations $T_{l_i}(a_i, b_i, k, p)$ $(i = 1, 2)$ of the rectifiable arcs in the generalized metric spaces is considered. Some sufficient conditions for the compatibility of these relations of the rectifiable arcs have been given here.

## Introduction

Let $E$ be an arbitrary non-empty set and let $l$ be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family $E_0$ of all non-empty subsets of the set $E$. The pair $(E, l)$ we shall call the generalized metric space (see [1]).

Let $a, b$ be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0 \tag{1}$$

By $S_l(p, r)_{a(r)}$ and $S_l(p, r)_{b(r)}$ (see [2]) we will denote in this paper the so-called $a(r), b(r)$-neighbourhoods of the sphere $S_l(p, r)$ in the space $(E, l)$.

We say that the pair $(A, B)$ of sets $A, B$ of the familiy $E_0$ is $(a, b)$-clustered at the point $p$ of the space $(E, l)$, if 0 is the cluster point of the set of all real numbers $r > 0$ such that the sets of the form $A \cap S_l(p, r)_{a(r)}$ and $B \cap S_l(p, r)_{a(r)}$ are non-empty.

Let (see [1, 4])

$$T_l(a, b, k, p) = \{(A, B) : \ A, B \in E_0, \text{ the pair } (A, B) \text{ is } (a, b)\text{-clustered at the point } p \text{ of the space } (E, l) \text{ and } \frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0\} \tag{2}$$

If $(A,B) \in T_l(a,b,k,p)$, then we say that the set $A \in E_0$ is $(a,b)$-tangent of order $k$ $(k > 0)$ to the set $B \in E_0$ at the point $p$ of the space $(E,l)$.

$T_l(a,b,k,p)$ defined by (2) we shall call the ralation of $(a,b)$-tangency of order $k$ of sets at the point $p$ of the generalized metric space $(E,l)$.

Two tangency relations of sets $T_{l_1}(a_1,b_1,k,p)$, $T_{l_2}(a_2,b_2,k,p)$ are called compatible if $(A,B) \in T_{l_1}(a_1,b_1,k,p)$ if and only if $(A,B) \in T_{l_2}(a_2,b_2,k,p)$ for $(A,B) \in E_0$.

Let $\rho$ be a metric of the set $E$ and let $A,B$ be arbitrary sets of the family $E_0$. Let us denote

$$\rho(A,B) = \inf\{\rho(x,y):\ x \in A,\ y \in B\}, \quad d_\rho A = \sup\{\rho(x,y):\ x,y \in A\} \quad (3)$$

By $\mathfrak{F}_\rho$ we denote the class of all functions $l$ fulfilling the conditions:

$1^0$ $l : E_0 \times E_0 \longrightarrow \langle 0,\infty)$,

$2^0$ $\rho(A,B) \le l(A,B) \le d_\rho(A \cup B)$ for $A,B \in E_0$.

Since

$$\rho(x,y) = \rho(\{x\},\{y\}) \le l(\{x\},\{y\}) \le d_\rho(\{x\} \cup \{y\}) = \rho(x,y),$$

then from here it follows that

$$l(\{x\},\{y\}) = \rho(x,y) \ \text{ for } \ l \in \mathfrak{F}_\rho \ \text{ and } \ x,y \in E \quad (4)$$

In the present paper the problem of the compatibility of the tangency relations of the rectifiable arcs in the spaces $(E,l)$, for the functions $l$ belonging to the class $\mathfrak{F}_\rho$ is considered.

## 1. The compatibility of the tangency relations of rectifiable arcs

Let $\rho$ be a metric of the set $E$, and let $A$ be any set of the family $E_0$. By $A'$ we shall denote the set of all cluster points of the set $A$.

Let $\widetilde{A}_p$ be the class of of the form (see papers [1, 4, 5]):

$\widetilde{A}_p = \{A \in E_0$: $A$ is rectifiable arc with the origin at the point $p \in E$ and

$$\lim_{A \ni x \to p} \frac{\ell(\widetilde{px})}{\rho(p,x)} = g < \infty\} \quad (5)$$

where $\ell(\widetilde{px})$ denotes the lenght of the arc $\widetilde{px}$ with the ends $p$ and $x$.

If $g = 1$, then we say that the arc $A \in E_0$ has the Archimedean property at the point $p$ of the metric space $(E,\rho)$, and is the arc of the class $A_p$.

We say (see [6]) that the set $A \in E_0$ has the Darboux property at the point $p$ of the metric space $(E, \rho)$, and we shall write this as: $A \in D_p(E, \rho)$, if there exists a number $\tau > 0$ such that $A \cap S_\rho(p, r) \neq \emptyset$ for $r \in (0, \tau)$.

Because any rectifiable arc $A$ with the origin at the point $p \in E$ has the Darboux property at the point $p$ of the metric space $(E, \rho)$, then from here and from the considerations of the papers [1, 4, 5] it follows the inclusion $\widetilde{A}_p \subset \widetilde{M}_p \cap D_p(E, \rho)$, where

$$\begin{gathered}\widetilde{M}_p = \{A \in E_0\colon\ p \in A' \text{ and there exists } \mu > 0 \text{ such that} \\ \text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \text{for every pair of points } (x, y) \in [A, p; \mu] \\ \text{if } \rho(p, x) < \delta \text{ and } \frac{\rho(x, A)}{\rho(p, x)} < \delta, \text{ then } \frac{\rho(x, y)}{\rho(p, x)} < \varepsilon\}\end{gathered} \tag{6}$$

and

$$[A, p; \mu] = \{(x, y)\colon\ x \in E,\ y \in A \text{ and } \mu\rho(x, A) < \rho(p, x) = \rho(p, y)\}.$$

Because $\widetilde{M}_p = \widetilde{M}_{p,1}$, then from here, from Theorem 2.1 of the paper [7] and from the above inclusion it follows the following corollary:

**Corollary 1.** *If in the metric space $(E, \rho)$ the arc $A$ belongs to the class $\widetilde{A}_p$, then*

$$\frac{a(r)}{r} \xrightarrow[r \to 0^+]{} 0 \tag{7}$$

*if and only if*

$$\frac{1}{r} d_\rho(A \cap S_\rho(p, r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{8}$$

Using this corollary we shall prove:

**Theorem 1.** *If $l_i \in \mathfrak{F}_\rho$ $(i = 1, 2)$,*

$$\frac{a(r)}{r} \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r} \xrightarrow[r \to 0^+]{} 0 \tag{9}$$

*then for arbitrary rectifiable arcs of the classes $\widetilde{A}_p$ the tangency relations $T_{l_1}(a, b, k, p)$ and $T_{l_2}(a, b, k, p)$ are compatible.*

`Proof.` Let us assume that $(A, B) \in T_{l_1}(a, b, 1, p)$ for $A, B \in \widetilde{A}_p$. Hence and from (6) it follows that the pair of sets $(A, B)$ is $(a, b)$-clustered at the point $p$ of the space $(E, l_1)$ and

$$\frac{1}{r} l_1(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{10}$$

From the inequality

$$d_\rho(A \cup B) \leq d_\rho A + d_\rho B + \rho(A,B) \quad \text{for} \quad A, B \in E_0 \tag{11}$$

from the properties of the function $f$ and from the fact that $l_1, l_2 \in \mathfrak{F}_\rho$ we get

$$\begin{aligned}
&\left| \frac{1}{r} l_2(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) - \frac{1}{r} l_1(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) \right| \\
&\qquad \leq \frac{1}{r} d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_\rho(p,r)_{b(r)})) \\
&\qquad\quad - \frac{1}{r} \rho(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) \\
&\qquad \leq \frac{1}{r} d_\rho(A \cap S_\rho(p,r)_{a(r)}) + \frac{1}{r} d_\rho(A \cap S_\rho(p,r)_{a(r)})
\end{aligned} \tag{12}$$

From the assumption (9) and from Corollary 1 it follows that

$$\frac{1}{r} d_\rho(A \cap S_\rho(p,r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{13}$$

and

$$\frac{1}{r} d_\rho(B \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{14}$$

From (13), (14) and from the inequality (12) we get

$$\frac{1}{r} l_2(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{15}$$

Since the functions $l_1, l_2 \in \mathfrak{F}_\rho$ generate on the set $E$ the same metric $\rho$ (see (6)), from the fact that the pair of sets $(A,B)$ is $(a,b)$-clustered at the point $p$ of the space $(E, l_1)$ it follows that is $(a,b)$-clustered at the point $p$ of the space $(E, l_2)$. Hence and from (15) it results that $(A,B) \in T_{l_2}(a,b,1,p)$.

If $(A,B) \in T_{l_2}(a,b,1,p)$, then similarly we prove that $(A,B) \in T_{l_1}(a,b,1,p)$. Hence it follows that the tangency relations $T_{l_1}(a,b,1,p)$ and $T_{l_2}(a,b,1,p)$ are compatible in the class $\widetilde{A}_p$ of rectifiable arcs.

Let $a_i, b_i$ $(i = 1,2)$ be non-negative real functions defined in a certain right-hand side neighbourhood of 0 and fulfilling the condition

$$a_i(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b_i(r) \xrightarrow[r \to 0^+]{} 0 \tag{16}$$

Let us denote

$$\breve{a} = \max(a_1, a_2), \quad \breve{b} = \max(b_1, b_2) \tag{17}$$

Now we prove the following theorem:

**Theorem 2.** *If* $l \in \mathfrak{F}_\rho$ *and*

$$\frac{a_i(r)}{r} \xrightarrow[r\to 0^+]{} 0, \quad \frac{b_i(r)}{r} \xrightarrow[r\to 0^+]{} 0 \quad \text{for} \quad i = 1, 2 \tag{18}$$

*then for arbitrary arcs of the class* $\widetilde{A}_p$ *the tangency relations* $T_l(a_1, b_1, 1, p)$ *and* $T_l(a_2, b_2, 1, p)$ *are compatible.*

`Proof.` Let us assume that $(A, B) \in T_l(a_1, b_1, 1, p)$ for any function $l \in \mathfrak{F}_\rho$ and sets $A, B \in \widetilde{A}_p$. Hence it follows that

$$\frac{1}{r} l(A \cap S_\rho(p, r)_{a_1(r)}, B \cap S_\rho(p, r)_{b_1(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{19}$$

From the inequality (11), from (17) and from the fact that $l \in \mathfrak{F}_\rho$ we get

$$\begin{aligned}
&\left| \frac{1}{r}(l(A \cap S_\rho(p, r)_{a_2(r)}, B \cap S_\rho(p, r)_{b_2(r)}) - l(A \cap S_\rho(p, r)_{a_1(r)}, B \cap S_\rho(p, r)_{b_1(r)})) \right| \\
&\le \frac{1}{r} d_\rho((A \cap S_\rho(p, r)_{a_2(r)}) \cup (B \cap S_\rho(p, r)_{b_2(r)})) \\
&\quad - \frac{1}{r} \rho(A \cap S_\rho(p, r)_{a_1(r)}, B \cap S_\rho(p, r)_{b_1(r)}) \\
&\le \frac{1}{r} d_\rho((A \cap S_\rho(p, r)_{\breve{a}(r)}) \cup (B \cap S_\rho(p, r)_{\breve{b}(r)})) \\
&\quad - \frac{1}{r} \rho(A \cap S_\rho(p, r)_{\breve{a}(r)}, B \cap S_\rho(p, r)_{\breve{b}(r)}) \\
&\le \frac{1}{r} d_\rho(A \cap S_\rho(p, r)_{\breve{a}(r)}) + \frac{1}{r} d_\rho(B \cap S_\rho(p, r)_{\breve{b}(r)})
\end{aligned} \tag{20}$$

From (17), (18) and from Corollary 1 it follows that

$$\frac{1}{r} d_\rho((A \cap S_\rho(p, r)_{\breve{a}(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{21}$$

and

$$\frac{1}{r} d_\rho(B \cap S_\rho(p, r)_{\breve{b}(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{22}$$

From (19), (21), (22) and from the inequality (20) we have

$$\frac{1}{r} l(A \cap S_\rho(p, r)_{a_2(r)}, B \cap S_\rho(p, r)_{b_2(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{23}$$

From the fact that the sets $A, B \in D_p(E, \rho)$ it follows that the pair of sets $(A, B)$ is $(a, b)$- clustered at the point $p$ of the metric space $(E, \rho)$. Hence and from (23) we obtain that $(A, B) \in T_l(a_2, b_2, 1, p)$.

If the pair of sets $(A, B) \in T_l(a_2, b_2, 1, p)$, then identically we prove that $(A, B) \in T_l(a_1, b_1, 1, p)$. From here it follows that the tangency relations $T_l(a_1, b_1, 1, p)$, $T_l(a_2, b_2, 1, p)$ are compatible in the class $\widetilde{A}_p$ of rectifiable arcs.

From the Theorems 1 and 2 it follows:

**Corollary 2.** *If* $l_i \in \mathfrak{F}_\rho$ *and the functions* $a_i, b_i$ $(i = 1, 2)$ *fulfil the condition* (18)*, then the tangency relations* $T_{l_1}(a_1, b_1, 1, p)$ *and* $T_{l_2}(a_2, b_2, 1, p)$ *are compatible in the class* $\widetilde{A}_p$ *of rectifiable arcs.*

All results presented in this paper are true for the rectifiable arcs of the class $A_p$ having the Archimedean property at the point $p$ of the metric space $(E, \rho)$.

## References


[1] Waliszewski W., On the tangency of sets in generalized metric spaces, Ann. Polon. Math. 1973, 28, 275-284.

[2] Konik T., On the reflexivity symmetry and transitivity of the tangency relations of sets of the class $\widetilde{M}_{p,k}$, J. Geom. 1995, 52, 142-151.

[3] Konik T., On some tangency relation of sets, Publ. Math. Debrecen 1999, 55/3-4, 411-419.

[4] Chądzyńska A., On some classes of sets related to the symmetry of the tangency relation in a metric space, Ann. Soc. Math. Polon., Comm. Math. 1972, 16, 219-228.

[5] Waliszewski W., On the tangency of sets in a metric space, Colloq. Math. 1966, 15, 127-131.

[6] Konik T., On some property of the tangency relation of sets, Balkan Journal of Geometry and Its Applications 2007, 12(1), 76-84.

[7] Konik T., On the sets of the classes $\widetilde{M}_{p,k}$, Demonstratio Math. 2000, 33(2), 407-417.

[8] Gołąb S., Moszner Z., Sur le contact des courbes dans les espaces metriques généraux, Colloq. Math. 1963, 10, 105-311.

[9] Pascali E., Tangency and ortogonality in metric spaces, Demonstratio Math. 2005, 38(2), 437-449.