ON THE COMATIBILITY OF THE TANGENCY RELATIONS OF RECTIFIABLE ARCS

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Abstract. In this paper the problem of the compatibility of the tangency relations $T_i(a, b, k, p)$ ($i = 1, 2$) of the rectifiable arcs in the generalized metric spaces is considered. Some sufficient conditions for the compatibility of these relations of the rectifiable arcs have been given here.

Introduction

Let $E$ be an arbitrary non-empty set and let $l$ be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family $E_0$ of all non-empty subsets of the set $E$. The pair $(E, l)$ we shall call the generalized metric space (see [1]).

Let $a, b$ be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

\[ a(r) \to 0 \quad \text{and} \quad b(r) \to 0 \quad r \to 0^+ \]  

(1)

By $S_l(p, r)_{a(r)}$ and $S_l(p, r)_{b(r)}$ (see [2]) we will denote in this paper the so-called $a(r)$, $b(r)$-neighbourhoods of the sphere $S_l(p, r)$ in the space $(E, l)$.

We say that the pair $(A, B)$ of sets $A, B$ of the family $E_0$ is $(a, b)$-clustered at the point $p$ of the space $(E, l)$, if 0 is the cluster point of the set of all real numbers $r > 0$ such that the sets of the form $A \cap S_l(p, r)_{a(r)}$ and $B \cap S_l(p, r)_{b(r)}$ are non-empty.

Let (see [1, 4])

\[ T_i(a, b, k, p) = \{(A, B) : A, B \in E_0, \text{ the pair } (A, B) \text{ is } (a, b)\text{-clustered} \] 

at the point $p$ of the space $(E, l)$ and

\[ \frac{1}{p^k}(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \to 0 \quad r \to 0^+ \]  

(2)
If \((A, B) \in T_l(a, b, k, p)\), then we say that the set \(A \in E_0\) is \((a, b)\)-tangent of order \(k\) \((k > 0)\) to the set \(B \in E_0\) at the point \(p\) of the space \((E, l)\).

\(T_l(a, b, k, p)\) defined by (2) we shall call the relation of \((a, b)\)-tangency of order \(k\) of sets at the point \(p\) of the generalized metric space \((E, l)\).

Two tangency relations of sets \(T_{l_1}(a_1, b_1, k, p), T_{l_2}(a_2, b_2, k, p)\) are called compatible if \((A, B) \in T_{l_1}(a_1, b_1, k, p)\) if and only if \((A, B) \in T_{l_2}(a_2, b_2, k, p)\) for \((A, B) \in E_0\).

Let \(\rho\) be a metric of the set \(E\) and let \(A, B\) be arbitrary sets of the family \(E_0\). Let us denote

\[\rho(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}, \quad d_\rho A = \sup\{\rho(x, y) : x, y \in A\}\]  

(3)

By \(\mathfrak{F}_\rho\) we denote the class of all functions \(l\) fulfilling the conditions:

1. \(l : E_0 \times E_0 \to (0, \infty)\),
2. \(\rho(A, B) \leq l(A, B) \leq d_\rho(A \cup B)\) for \(A, B \in E_0\).

Since

\[\rho(x, y) = \rho(\{x\}, \{y\}) \leq l(\{x\}, \{y\}) \leq d_\rho(\{x\} \cup \{y\}) = \rho(x, y),\]

then from here it follows that

\[l(\{x\}, \{y\}) = \rho(x, y)\]  

for \(l \in \mathfrak{F}_\rho\) and \(x, y \in E\)  

(4)

In the present paper the problem of the compatibility of the tangency relations of the rectifiable arcs in the spaces \((E, l)\), for the functions \(l\) belonging to the class \(\mathfrak{F}_\rho\) is considered.

1. The compatibility of the tangency relations of rectifiable arcs

Let \(\rho\) be a metric of the set \(E\), and let \(A\) be any set of the family \(E_0\). By \(A'\) we shall denote the set of all cluster points of the set \(A\).

Let \(\tilde{A}_p\) be the class of of the form (see papers [1, 4, 5]):

\[\tilde{A}_p = \{ A \in E_0: A\text{ is rectifiable arc with the origin at the point } p \in E \text{ and }\]

\[\lim_{A \to p} \frac{\ell(p \bar{x})}{\rho(p, x)} = g < \infty\}

(5)

where \(\ell(p \bar{x})\) denotes the length of the arc \(p \bar{x}\) with the ends \(p\) and \(x\).

If \(g = 1\), then we say that the arc \(A \in E_0\) has the Archimedean property at the point \(p\) of the metric space \((E, \rho)\), and is the arc of the class \(A_p\).
We say (see [6]) that the set $A \in E_0$ has the Darboux property at the point $p$ of the metric space $(E, \rho)$, and we shall write this as: $A \in D_p(E, \rho)$, if there exists a number $\tau > 0$ such that $A \cap S_\rho(p,r) \neq \emptyset$ for $r \in (0, \tau)$.

Because any rectifiable arc $A$ with the origin at the point $p \in E$ has the Darboux property at the point $p$ of the metric space $(E, \rho)$, then from here and from the considerations of the papers [1, 4, 5] it follows the inclusion $\widetilde{A}_p \subset \widetilde{M}_p \cap D_p(E, \rho)$, where

$$\widetilde{M}_p = \{ A \in E_0 : p \in A' \text{ and there exists } \mu > 0 \text{ such that}$$

for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{for every pair of points } (x, y) \in [A, p; \mu]$$

if $\rho(p, x) < \delta$ and $\frac{\rho(x, A)}{\rho(p, x)} < \delta$, then $\frac{\rho(x, y)}{\rho(p, x)} < \varepsilon$ \hspace{1cm} (6)

and

$$[A, p; \mu] = \{ (x, y) : x \in E, y \in A \text{ and } \mu \rho(x, A) < \rho(p, x) = \rho(p, y) \}.$$ 

Because $\widetilde{M}_p = \widetilde{M}_{p,1}$, then from here, from Theorem 2.1 of the paper [7] and from the above inclusion it follows the following corollary:

**Corollary 1.** If in the metric space $(E, \rho)$ the arc $A$ belongs to the class $\widetilde{A}_p$, then

$$\frac{a(r)}{r} \underset{r \rightarrow 0^+}{\longrightarrow} 0$$

if and only if

$$\frac{1}{r} d_\rho(A \cap S_\rho(p, r)_{a(r)}) \underset{r \rightarrow 0^+}{\longrightarrow} 0$$

(8)

Using this corollary we shall prove:

**Theorem 1.** If $l_i \in \mathcal{F}_\rho$ $(i = 1, 2)$,

$$\frac{a(r)}{r} \underset{r \rightarrow 0^+}{\longrightarrow} 0 \text{ and } \frac{b(r)}{r} \underset{r \rightarrow 0^+}{\longrightarrow} 0$$

then for arbitrary rectifiable arcs of the classes $\widetilde{A}_p$ the tangency relations $T_{l_1}(a, b, k, p)$ and $T_{l_2}(a, b, k, p)$ are compatible.

**Proof.** Let us assume that $(A, B) \in T_{l_1}(a, b, 1, p)$ for $A, B \in \widetilde{A}_p$. Hence and from (6) it follows that the pair of sets $(A, B)$ is $(a, b)$-clustered at the point $p$ of the space $(E, l_1)$ and
\[
\frac{1}{r}l_1(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)}) \longrightarrow 0 \quad r \to 0^+
\]  

(10)

From the inequality

\[d_\rho(A \cup B) \leq d_\rho A + d_\rho B + \rho(A, B) \quad \text{for} \quad A, B \in E_0\]  

(11)

from the properties of the function \(f\) and from the fact that \(l_1, l_2 \in \mathcal{F}_\rho\) we get

\[
\left| \frac{1}{r}l_2(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)}) - \frac{1}{r}l_1(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)}) \right| \\
\leq \frac{1}{r}d_\rho((A \cap S_l(p, r)_{a(r)}) \cup (B \cap S_\rho(p, r)_{b(r)})) \\
- \frac{1}{r}d_\rho(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)}) \\
\leq \frac{1}{r}d_\rho(A \cap S_\rho(p, r)_{a(r)}) + \frac{1}{r}d_\rho(A \cap S_\rho(p, r)_{a(r)})
\]  

(12)

From the assumption (9) and from Corollary 1 it follows that

\[
\frac{1}{r}d_\rho(A \cap S_\rho(p, r)_{a(r)}) \longrightarrow 0 \quad r \to 0^+
\]  

(13)

and

\[
\frac{1}{r}d_\rho(B \cap S_\rho(p, r)_{b(r)}) \longrightarrow 0 \quad r \to 0^+
\]  

(14)

From (13), (14) and from the inequality (12) we get

\[
\frac{1}{r}l_2(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)}) \longrightarrow 0 \quad r \to 0^+
\]  

(15)

Since the functions \(l_1, l_2 \in \mathcal{F}_\rho\) generate on the set \(E\) the same metric \(\rho\) (see (6)), from the fact that the pair of sets \((A, B)\) is \((a, b)\)-clustered at the point \(p\) of the space \((E, l_1)\) it follows that is \((a, b)\)-clustered at the point \(p\) of the space \((E, l_2)\). Hence and from (15) it results that \((A, B) \in T_{l_2}(a, b, 1, p)\).

If \((A, B) \in T_{l_2}(a, b, 1, p)\), then similarly we prove that \((A, B) \in T_{l_1}(a, b, 1, p)\). Hence it follows that the tangency relations \(T_{l_1}(a, b, 1, p)\) and \(T_{l_2}(a, b, 1, p)\) are compatible in the class \(A_p\) of rectifiable arcs.

Let \(a_i, b_i\) \((i = 1, 2)\) be non-negative real functions defined in a certain right-hand side neighbourhood of 0 and fulfilling the condition

\[
a_i(r) \longrightarrow 0 \quad \text{and} \quad b_i(r) \longrightarrow 0 \quad r \to 0^+
\]  

(16)

Let us denote
\[ \ddot{a} = \max(a_1, a_2), \quad \ddot{b} = \max(b_1, b_2) \] (17)

Now we prove the following theorem:

**Theorem 2.** If \( l \in \mathcal{F}_\rho \) and

\[
\frac{a_i(r)}{r} \xrightarrow{r \to 0^+} 0, \quad \frac{b_i(r)}{r} \xrightarrow{r \to 0^+} 0 \quad \text{for} \quad i = 1, 2
\] (18)

then for arbitrary arcs of the class \( \tilde{A}_p \) the tangency relations \( T_l(a_1, b_1, 1, p) \) and \( T_l(a_2, b_2, 1, p) \) are compatible.

**Proof.** Let us assume that \((A, B) \in T_l(a_1, b_1, 1, p)\) for any function \( l \in \mathcal{F}_\rho \) and sets \( A, B \in \tilde{A}_p \). Hence it follows that

\[
\frac{1}{r}(A \cap S_\rho(p, r)_{a_1(r)}, B \cap S_\rho(p, r)_{b_1(r)}) \xrightarrow{r \to 0^+} 0
\] (19)

From the inequality (11), from (17) and from the fact that \( l \in \mathcal{F}_\rho \) we get

\[
\left| \frac{1}{r}(l(A \cap S_\rho(p, r)_{a_2(r)}, B \cap S_\rho(p, r)_{b_2(r)}) - l(A \cap S_\rho(p, r)_{a_1(r)}, B \cap S_\rho(p, r)_{b_1(r)})) \right|
\leq \frac{1}{r}d_\rho((A \cap S_\rho(p, r)_{a_2(r)}) \cup (B \cap S_\rho(p, r)_{b_2(r)}))
- \frac{1}{r}\rho(A \cap S_\rho(p, r)_{a_1(r)}, B \cap S_\rho(p, r)_{b_1(r)})
\leq \frac{1}{r}d_\rho((A \cap S_\rho(p, r)_{\ddot{a}(r)}) \cup (B \cap S_\rho(p, r)_{\ddot{b}(r)}))
- \frac{1}{r}\rho(A \cap S_\rho(p, r)_{\ddot{a}(r)}, B \cap S_\rho(p, r)_{\ddot{b}(r)})
\leq \frac{1}{r}d_\rho(A \cap S_\rho(p, r)_{\ddot{a}(r)}) + \frac{1}{r}d_\rho(B \cap S_\rho(p, r)_{\ddot{b}(r)})
\] (20)

From (17), (18) and from Corollary 1 it follows that

\[
\frac{1}{r}d_\rho((A \cap S_\rho(p, r)_{\ddot{a}(r)}) \xrightarrow{r \to 0^+} 0
\] (21)

and

\[
\frac{1}{r}d_\rho(B \cap S_\rho(p, r)_{\ddot{b}(r)}) \xrightarrow{r \to 0^+} 0
\] (22)

From (19), (21), (22) and from the inequality (20) we have

\[
\frac{1}{r}(l(A \cap S_\rho(p, r)_{a_2(r)}, B \cap S_\rho(p, r)_{b_2(r)}) \xrightarrow{r \to 0^+} 0
\] (23)
From the fact that the sets $A, B \in D_p(E, \rho)$ it follows that the pair of sets $(A, B)$ is $(a, b)$-clustered at the point $p$ of the metric space $(E, \rho)$. Hence and from (23) we obtain that $(A, B) \in T_i(a_2, b_2, 1, p)$.

If the pair of sets $(A, B) \in T_i(a_2, b_2, 1, p)$, then identically we prove that $(A, B) \in T_i(a_1, b_1, 1, p)$. From here it follows that the tangency relations $T_i(a_1, b_1, 1, p), T_i(a_2, b_2, 1, p)$ are compatible in the class $\tilde{A}_p$ of rectifiable arcs.

From the Theorems 1 and 2 it follows:

**Corollary 2.** If $l_i \in \tilde{Z}_p$ and the functions $a_i, b_i$ ($i = 1, 2$) fulfil the condition (18), then the tangency relations $T_i(a_1, b_1, 1, p)$ and $T_i(a_2, b_2, 1, p)$ are compatible in the class $\tilde{A}_p$ of rectifiable arcs.

All results presented in this paper are true for the rectifiable arcs of the class $A_p$ having the Archimedean property at the point $p$ of the metric space $(E, \rho)$.

**References**